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THE NON-COMPACTNESS OF SQUARE

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§1. Introduction. This note proves two theorems. The first is that it is consistent to have $\square_{\omega_n}$ for every n, but not have $\square_{\aleph_\omega}$. This is done by carefully collapsing a supercompact cardinal and adding square sequences to each ω_n. The crux of the proof is that in the resulting model every stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega)$ *reflects* to an ordinal of cofinality ω_1, that is to say it has stationary intersection with such an ordinal.

This result contrasts with compactness properties of square shown in [3]. In that paper it is shown that if one has square at every ω_n, then there is a square type sequence on the points of cofinality ω_k, $k > 1$ in $\aleph_{\omega+1}$. In particular at points of cofinality greater than ω_1 there is a strongly non-reflecting stationary set of points of countable cofinality.

The second result answers a question of Džamonja, by showing that there can be no squarelike sequence above a supercompact cardinal, where "squarelike" means that one replaces the requirement that the cofinal sets be closed and unbounded by the requirement that they be stationary at all points of uncountable cofinality.

§2. Some lemmas. In this section we define a forcing notion and show some lemmas. The forcing notion is a standard style of Namba forcing and the lemmas are standard. We prove them here for the benefit of the reader.

Let $n \to (n_0, n_1)$ be a bijection from ω to $\omega \times (\omega \backslash \{0, 1\})$. We say that a tree is *standard* for our partial ordering iff

1. $T \subset (\aleph_\omega)^{<\omega}$.
2. For all $\sigma \in T$ and $n \in \mathrm{dom}(\sigma), \sigma(n) \in \omega_{n_1}$.
3. There is a maximal $\sigma \in T$ (called the *stem* of T) such that for all $\tau \in T, \sigma \subset \tau$ or $\tau \subset \sigma$.
4. For all τ extending the stem σ, if $n \in \mathrm{dom}(\tau), n \supset \mathrm{dom}(\sigma)$ then $|\{\alpha : \tau \restriction n^\frown \alpha \in T\}| = \omega_{n_1}$.

Our partial ordering $\mathscr{P}$ will consist of standard trees with the ordering of inclusion.

It is easy to verify the following facts:

- $|\mathscr{P}| = |\prod_{n\in\omega} 2^{\omega_n}|$.

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