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OUTER MODELS AND GENERICITY

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§1. Introduction. Why is forcing the only known method for constructing outer models of set theory?

If V is a standard transitive model of ZFC, then a standard transitive model W of ZFC is an ***outer model*** of V if $V \subseteq W$ and $V \cap \mathrm{OR} = W \cap \mathrm{OR}$.

Is every outer model of a given model a generic extension? At one point Solovay conjectured that if $0^{\#}$ exists, then every real that does not construct $0^{\#}$ lies in $L[G]$, for some G that is generic for some forcing $\mathbb{P} \in L$. Famously, this was refuted by Jensen's coding theorem. He produced a real that is generic for an L-definable class forcing property, but does not lie in any set forcing extension of L.

Beller, Jensen, and Welch in *Coding the universe* [BJW] revived Solovay's conjecture by asking the following question: *Let $a \subseteq \omega$ be such that $L[a] \models$ "$0^{\#}$ does not exist". Is there a $b \in L[a]$ such that $a \notin L[b]$ and a is set generic over $L[b]$.*

In [S1] it was shown that even if arbitrary inner models are allowed, rather than just ones of the form $L[b]$, and even if we allow a to be class generic, the answer is No in general:

THEOREM 1.1. *Let L_α be a minimal countable standard transitive model of ZFC. There exists a real x_{nwg} having the following three properties:*

(1) $x_{nwg} \notin L_\alpha$.
(2) $L_\alpha[x_{nwg}] \models \mathrm{ZFC}$.
(3) *x_{nwg} is not definably generic over any outer model of L_α that does not already contain x_{nwg}.*

A precise statement of (3) is the following: Assume that V is an outer model of L_α and that $\mathbb{P}$ is a V-amenable partial ordering such that $(V; \mathbb{P})$ satisfies ZFC. Assume that the forcing relation restricted to sentences of bounded complexity is definable over $(V; \mathbb{P})$. (See Remark 1.8 regarding this hypothesis.) If G is a maximal filter on $\mathbb{P}$ meeting every dense subclass of $\mathbb{P}$ that is definable over $(V; \mathbb{P})$ and $x_{\mathrm{nwg}} \in V[G]$, then $x_{\mathrm{nwg}} \in V$.

For the sake of clarity, an elementary remark is in order. As is customary, we write "V" for the standard structure $(V; \in)$. If $S \subseteq V$, then "$(V; S)$ satisfies ZFC" means that $(V; S, \in)$ satisfies ZFC in an enlarged language with a predicate symbol for S. In this case the axioms of ZFC are augmented by instances of Collection and Separation formulated in the enlarged language. We call this extended theory

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