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SOME STRONGLY UNDECIDABLE NATURAL ARITHMETICAL PROBLEMS, WITH AN APPLICATION TO INTUITIONISTIC THEORIES

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§1. Introduction. Although Church and Turing presented their path-breaking undecidability results immediately after their explication of effective decidability in 1936, it has been generally felt that these results do not have any direct bearing on ordinary mathematics but only contribute to logic, metamathematics and the theory of computability. Therefore it was such a celebrated achievement when Yuri Matiyasevich in 1970 demonstrated that the problem of the solvability of Diophantine equations is undecidable. His work was building essentially on the earlier work by Julia Robinson, Martin Davis and Hilary Putnam (1961), who had showed that the problem of solvability of exponential Diophantine equations is undecidable. One should note, however, that although it was only Matiyasevich's result which finally solved Hilbert's tenth problem, already the earlier result had provided a perfectly natural problem of ordinary number theory which is undecidable.[1]

Nevertheless, both the set of Diophantine equations with solutions and the set of exponential Diophantine equations with solutions are still semi-decidable, that is, recursively enumerable (i.e., Σ^0_1); if an equation in fact has a solution, this can be eventually verified. More generally, they are — as are their complements, the sets of equations with no solutions, which are Π^0_1 — also Trial and Error decidable (Putnam [1965]), or decidable in the limit (Shoenfield [1959]), for every Δ^0_2 set is (and conversely). This last-mentioned natural "liberalized" notion of decidability has begun more recently to play an essential role e.g., in so-called Formal Learning Theory (see e.g., Osherson, Stob, and Weinstein [1986], Kelly [1996]).[2]

Later, the researchers in Diophantine decision problems have studied various problems related to the cardinality of solutions (see Davis [1972], Davis, Putnam, and Robinson [1976], Smoryński [1977]; cf. Davis [1973], [1977], Matiyasevich [1993], Smoryński [1991]). But to date the strongest results explicitly presented in

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[1]It should be added that these were not the first problems from ordinary mathematics which were shown to be undecidable. There were, for example, various problems related to the theory of groups which had been shown to be undecidable already earlier. For a good overview, see Davis [1977]. However, the problems of Matiyasevich and Robinson, Davis and Putnam were arguably unique in their simplicity and elementary nature, and in any case the first natural undecidable problems from ordinary arithmetic.

[2]Note also that by Post's Theorem, every Δ^0_2 set is recursive in some Σ^0_1 or some Π^0_1 set, and consequently, is recursive in a Σ^0_1 complete set such as the Halting set K_0. This also confirms my view that no Δ^0_2 set is "strongly undecidable" but "decidable in a weak sense" (being decidable relative to a semi-decidable set), and that in order to be really "strongly undecidable" a set must be beyond Δ^0_2.

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