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INTERNAL CONSISTENCY AND THE INNER MODEL HYPOTHESIS

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There are two standard ways to establish consistency in set theory. One is to prove consistency using *inner models*, in the way that Gödel proved the consistency of GCH using the inner model L. The other is to prove consistency using *outer models*, in the way that Cohen proved the consistency of the negation of CH by enlarging L to a forcing extension $L[G]$.

But we can demand more from the outer model method, and we illustrate this by examining Easton's strengthening of Cohen's result:

THEOREM 1 (Easton's Theorem). *There is a forcing extension $L[G]$ of L in which GCH fails at every regular cardinal.*

Assume that the universe V of all sets is rich in the sense that it contains inner models with large cardinals. Then what is the relationship between Easton's model $L[G]$ and V? In particular, are these models *compatible*, in the sense that they are inner models of a common third model? If not, then the failure of GCH at every regular cardinal is consistent only in a weak sense, as it can only hold in universes which are incompatible with the universe of all sets. Ideally, we would like $L[G]$ to not only be compatible with V, but to be an inner model of V.

We say that a statement is *internally consistent* iff it holds in some inner model, under the assumption that there are inner models with large cardinals. By specifying what large cardinals are required, we obtain a new type of consistency result. Let $\text{Con}(\text{ZFC} + \varphi)$ stand for "$\text{ZFC} + \varphi$ is consistent" and $\text{Icon}(\text{ZFC} + \varphi)$ stand for "there is an inner model of $\text{ZFC} + \varphi$". A typical consistency result takes the form

$$\text{Con}(\text{ZFC} + \text{LC}) \rightarrow \text{Con}(\text{ZFC} + \varphi)$$

where LC denotes some large cardinal axiom. An *internal* consistency result takes the form

$$\text{Icon}(\text{ZFC} + \text{LC}) \rightarrow \text{Icon}(\text{ZFC} + \varphi).$$

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