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PARTITION THEOREMS AND COMPUTABILITY THEORY

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§1. Introduction. The connections between mathematical logic and combinatorics have a rich history. This paper focuses on one aspect of this relationship: understanding the strength, measured using the tools of computability theory and reverse mathematics, of various partition theorems. To set the stage, recall two of the most fundamental combinatorial principles, König's Lemma and Ramsey's Theorem. We denote the set of natural numbers by ω and the set of finite sequences of natural numbers by $\omega^{<\omega}$. We also identify each $n \in \omega$ with its set of predecessors, so $n = \{0, 1, 2, \dots, n-1\}$.

DEFINITION 1.1.

1. A *tree* is a subset T of $\omega^{<\omega}$ such that for all $\sigma \in T$, if $\tau \in \omega^{<\omega}$ and $\tau \subseteq \sigma$, then $\tau \in T$.
2. If T is a tree and $S \subseteq T$ is also a tree, we say that S is a *subtree* of T.
3. A tree T is *bounded* if there exists $h\colon \omega \to \omega$ such that for all $\sigma \in T$ and $k \in \omega$ with $|\sigma| > k$, we have $\sigma(k) \leq h(k)$.
4. A *branch* of a tree T is a function $f : \omega \to \omega$ such that $f \upharpoonright n \in T$ for all $n \in \omega$.

THEOREM 1.2 (König's Lemma). *Every infinite bounded tree has a branch.*

DEFINITION 1.3.

1. Given a set $Z \subseteq \omega$ and $n \in \omega$, we let $[Z]^n = \{x \subseteq Z : |x| = n\}$.
2. Suppose that $n, p \geq 1$ and $f : [\omega]^n \to p$. Such an f is called a *p-coloring* of $[\omega]^n$ and n is called the *exponent*. We say that a set $H \subseteq \omega$ is *homogeneous* for f if H is infinite and $f(x) = f(y)$ for all $x, y \in [H]^n$.

THEOREM 1.4 (Ramsey's Theorem [20]). *Suppose that* $n,\ p \geq 1$ *and* $f : [\omega]^n \to p$. *There exists a set* H *homogeneous for* f.

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