TSUKUBA J. MATH.
Vol. 31 No. 2 (2007), 253–259

A FURTHER NOTE ON THE GENERALIZED JOSEPHUS PROBLEM

By

Saburô Uchiyama

1. In our previous papers [1] and [2] we have observed several interesting and significant aspects of the generalized Josephus problem. In the present article we shall again concern ourselves with this problem. Thus, given a total number $n \geq 1$ and certain n objects numbered from 1 to n, and another integer $m \geq 1$, called the reduction coefficient, we arrange these n objects in a circle and, starting with the object numbered 1, and counting each object in turn around the circle, we eliminate every mth object until all of them are removed. By $a_m(k,n)$ $(1 \leq k \leq n)$ we denote as before the kth Josephus number, that is, the object number to be removed in the kth step of elimination. It is evident that we have

$$1 \leq a_m(k,n) \leq n \tag{1}$$

and

$$a_m(1,n) \equiv m \pmod{n}, \tag{2}$$

and that

$$a_m(k+1,n+1) \equiv a_m(1,n+1) + a_m(k,n) \pmod{n+1},$$

from which follows at once

$$a_m(k+1,n+1) \equiv m + a_m(k,n) \pmod{n+1} \tag{3}$$

in view of (2); (3) is the fundamental relation due to P. G. Tait for the Josephus numbers $a_m(k,n)$ (cf. [1; §§1–2]). In effect, the Josephus numbers $a_m(k,n)$ $(1 \leq k \leq n)$ are completely determined by the conditions (1), (2) and (3).

In what follows we devote ourselves to the study of the special case of $k = n$ and write for simplicity's sake $d_m(n) = a_m(n,n)$ as in [1]. We have then $d_m(1) = 1$ for any $m \geq 1$, and the fundamental relation (3) becomes

$$d_m(n+1) \equiv m + d_m(n) \pmod{n+1}. \tag{4}$$

Received March 9, 2006.