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ON PRIMES IN ARITHMETIC PROGRESSIONS

In honorem Saburô Uchiyama annos LXX nati

By

Hiroshi MIKAWA

1. Introduction

The Dirichlet theorem says that, for any coprime integers q and a, there are infinitely many primes which are congruent to a modulo q. See [16, Kap. IV], for instance. Then, for $(q,a)=1$, let $P(q,a)$ be the least prime in an arithmetic progression $p \equiv a \pmod{q}$. The extended Riemann hypothesis gives that

$$P(q,a) \ll q^{2+\varepsilon} \tag{1}$$

for any $\varepsilon > 0$. However it is conjectured that this exponent 2 could be replaced by 1.

The Linnik theorem unconditionally shows that

$$P(q,a) \ll q^{L}$$

with some absolute constant L, vide [16, Kap. X]. Many works have been done to obtain an explicit value of this Linnik constant. The best known result is $L = 5.5$ due to D. R. Heath-Brown [14].

The Bombieri-Vinogradov theorem, see [7, § 28], has the same power as the extended Riemann hypothesis in some sense. Indeed, it yields (1) for any given $a \neq 0$ and almost all q. In 1980 E. Fouvry and H. Iwaniec [10, 11] made a significant step beyond the extended Riemann hypothesis. Their ideas have been surprisingly developed by E. Fouvry [8, 9] and E. Bombieri, J. B. Friedlander and H. Iwaniec [4, 5]. In particular, it follows from [5] that, for any fixed $a \neq 0$ and almost all q,

$$P(q,a) \ll q^{2-\delta} \tag{2}$$

where $0 < \delta = \delta(q) \to 0$ as $q \to \infty$.

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