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THE SUM OF CONSECUTIVE FRACTIONAL PARTS

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1. Introduction

If we sum any h consecutive terms in the sequence

$$\{\theta\}, \{2\theta\}, \{3\theta\}, \ldots,$$

what are reasonable bounds for the sum? Of course, it is at least as large as 0 and no more than h. But that is far too too rough. Thus we consider the following problem:

What is the least upper bound and the greatest lower bound of

$$B_h(\theta) = \sum_{i=1}^{h} \{(N+i)\theta\}?$$

Here, θ is a given irrational number, and h is some positive integer. The variable N is restricted to the non-negative integers.

In the case of just one fractional part the bounds are known. As remarked in [2] we have

THEOREM 1. *Let the continued fraction expansion of θ be*

$$\theta = [a_0, a_1, a_2, \ldots],$$

and denote by $q_n (= a_n q_{n-1} + q_{n-2})$ the denominator of the nth convergent

$$[a_0, a_1, \ldots, a_n].$$

Then if q is an integer satisfying $0 < q < q_n$,

$$\begin{cases} \{q_{n-1}\theta\} \le \{q\theta\} \le \{(q_n - q_{n-1})\theta\} & \text{when } n \text{ is odd;} \\ \{(q_n - q_{n-1})\theta\} \le \{q\theta\} \le \{q_{n-1}\theta\} & \text{when } n \text{ is even.} \end{cases}$$

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