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CURVATURE BOUND AND TRAJECTORIES FOR MAGNETIC FIELDS ON A HADAMARD SURFACE

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Introduction.

On a complete oriented Riemannian manifold M, a closed 2-form $\boldsymbol{B}$ is called a *magnetic field*. Let Ω denote the skew symmetric operator $\Omega : \mathrm{TM} \to \mathrm{TM}$ defined by $\langle u, \Omega(v) \rangle = \boldsymbol{B}(u, v)$ for every u, $v \in TM$. We call a smooth curve γ a *trajectory* for $\boldsymbol{B}$ if it satisfies the equation $\nabla_{\dot{\gamma}} \dot{\gamma} = \Omega(\dot{\gamma})$. Since Ω is skew symmetric, we find that every trajectory has constant speed and is defined for $-\infty < t < \infty$. We shall call a trajectory *normal* if it is parametrized by its arc length. When γ is a trajectory for $\boldsymbol{B}$, the curve σ defined by $\sigma(t) = \gamma(\lambda t)$ with some constant λ is a trajectory for $\lambda\boldsymbol{B}$. We call the norm $\|\boldsymbol{B}_x\|$ of the operator $\boldsymbol{B}_x : T_xM \times T_xM \to \boldsymbol{R}$ the *strength* of the magnetic field at the point x. For the trivial magnetic field $\boldsymbol{B} = 0$, the case without the force of a magnetic field, trajectories are nothing but geodesics. In term of physics it is a trajectory of a charged particle under the action of the magnetic field. For a classical treatment and physical meaning of magnetic fields see [8].

On a Riemann surface M we can write down $\boldsymbol{B} = f \cdot \mathrm{Vol}_M$ with a smooth function f and the volum form Vol_M on M. When f is a constant function, the case of constant strength, the magnetic field is called *uniform*. On surfaces of constant curvature the feature of trajectories are well-known for every uniform magnetic field $k \cdot \mathrm{Vol}_M$. On a Euclidean plane $\boldsymbol{R}^2$ they are circles (in usual sense of Euclidean geometry) of radius $1/|k|$. On a sphere $S^2(c)$ they are small circles with prime period $2\pi/\sqrt{k^2+c}$. In these cases all trajectories are closed. On a hyperbolic plane $H^2(-c)$ of constant curvature $-c$, the situation is different. In his paper [4] Comtet showed that the feature of trajectories changes according to the strength of a uniform magnetic field $k \cdot \mathrm{Vol}_M$. When the strength $|k|$ is greater than $\sqrt{c}$, normal trajectories are still closed, hence bounded, but if $|k| \le \sqrt{c}$ they are unbounded simple curves, in particular, if $|k| = \sqrt{c}$ they are horocycles. In the preceeding paper [2] we studied trajectories for Kähler magnetic fields $k \cdot \boldsymbol{B}_J$,

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