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ON RAMANUJAN SUMS ON ARITHMETICAL SEMIGROUPS

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1. Introduction.

Let $f: N\rightarrow C$ be an arithmetic function and let $f^{*}=\mu * f$ denote the Dirichlet convolution and the Möbius function μ, so that

$$f^{*}(n)=\sum_{d\mid n}\mu(d)f\left(\frac{n}{d}\right), \quad n\geqq 1. \tag{1.1}$$

Let

$$c_{q}(n)=\sum_{\substack{h=1\\(h,q)=1}}^{q}\exp\left(2\Pi i\frac{hn}{q}\right) \tag{1.2}$$

be the Ramanujan's trigonometric sum. A Ramanujan series is a series of the form

$$\sum_{q=1}^{\infty}a_{q}c_{q}(n) \tag{1.3}$$

where $c_{q}(n)$ is Ramanujan's sum and

$$a_{q}=\sum_{m=1}^{\infty}\frac{f^{*}(mq)}{mq}. \tag{1.4}$$

Important result concerning Ramanujan's expansions of certain arithmetical functions has been given by Delange [2]. He proved the following result:

THEOREM A. *If* $\sum_{n=1}^{\infty}\frac{2^{\omega(n)}}{n}|f^{*}(n)|<\infty$, *where* $\omega(n)$ *is the number of distinct prime divisors of* n, *then* $\sum_{q=1}^{\infty}|a_{q}c_{q}(n)|<\infty$ *for every* n *and* $\sum_{q=1}^{\infty}a_{q}c_{q}(n)=f(n)$.

In his proof, Delange used the inequality

$$\sum_{d\mid k}|c_{d}(n)|\leqq 2^{\omega(k)}n, \tag{1.5}$$

see [2; Lemma, p. 263] and conjectured [2, p. 264] that his Lemma is best possible.

In [3] we proved the following identity:

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