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A GENERALIZATION OF HEREDITY IDEALS

(Dedicated to Professor Manabu Harada on his 60th birthday)

By

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1. Introduction

Throughout this note, A stands for a basic left and right artinian ring, J its Jacobson radical and $\{e_1, \cdots, e_n\}$ the complete set of orthogonal primitive idempotents in A. Let c_{ij} denote the composition length of ${}_{e_iAe_i}e_iAe_j$ for $1 \leqq i, j \leqq n$. The matrix $C(A)=(c_{ij})$ is called the left Cartan matrix of A.

Does $\operatorname{gl\,dim} A < \infty$ imply $\det C(A)=1$? This problem has been partially settled by several authors (e.g., Zacharia [7], Wilson [6], Burgess et al. [2], Fuller and Zimmermann-Huisgen [5] and so on), but is still open. There is a way to reduce the size of the matrix $C(A)$. Namely, if $\operatorname{proj\,dim}_A Ae_1/Je_1 < \infty$ and $\operatorname{Ext}_A^k(Ae_1/Je_1, Ae_1/Je_1)=0$ for $k>0$, then $\operatorname{gl\,dim}(1-e_1)A(1-e_1) \leqq \operatorname{gl\,dim} A + \operatorname{proj\,dim}_A Je_1$ and $\det C((1-e_1)A(1-e_1)) = \det C(A)$. This reduction was effectively used by Zacharia [7] to show that $\operatorname{gl\,dim} A \leqq 2$ implies $\det C(A)=1$ (see also Burgess et al. [2]). Unfortunately, as will be seen, Zacharia's reduction is not necessarily applicable if $\operatorname{gl\,dim} A \geqq 3$.

The aim of this note is to provide another type of reduction. To do this, we will generalize the notion of a heredity ideal which was first introduced by Cline, Parshall and Scott [3]. We are interested in a two-sided ideal I of A such that $\det C(A/I) = \det C(A)$ (of course, we claim $\operatorname{gl\,dim} A/I < \infty$ whenever $\operatorname{gl\,dim} A < \infty$). We will show that the trace ideal of a certain left A-module enjoys this property. We will prove the following

THEOREM. *Let Q be a torsionless left A-module and I its trace ideal. Suppose the following conditions:*

(a) $D=\operatorname{End}_A(Q)$ *is a division ring,*

(b) *the evaluation map* $Q \otimes_D \operatorname{Hom}_A(Q, A) \to A$ *is monic.*

(c) $\operatorname{Tor}_k^A(\operatorname{Tr} Q, Q)=0$ *for* $k \geqq 2$, *where* Tr *is the transpose, and*

(d) $\operatorname{proj\,dim}_A Q < \infty$.

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