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NON-NORMAL NUMBERS TO DIFFERENT BASES AND THEIR HAUSDORFF DIMENSION

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0. Introduction.

The notion of normal numbers was first introduced by Emile Borel [3] in 1909. He considered the decimal expansion of real numbers in the unit interval to the base r and assuming that every digit of their decimal expansions is independent and also takes all possible values $0,1,\cdots$ and $r-1$ with equal probability, he proved that almost all real numbers are normal to the base r in the sense of Lebesgue measure.

For a real number ω, we denote $\{\omega\}$ the fractional part of ω defined by

$$\{\omega\}=\omega-[\omega],$$

where $[\cdot]$ is the Gauss' symbol, so that $\{\omega\}$ is contained in the unit interval $I_0=[0,1)$ for every real number ω. We consider the decimal expansion of $\{\omega\}$ to the base r:

$$\{\omega\}=\sum_{n=1}^{\infty}\frac{x_n(\omega)}{r^n}, \tag{1}$$

where $x_n(\omega)$ is the n-th digit of development of $\{\omega\}$ and takes one of the values in $R=\{0,1,\cdots,r-1\}$. For an r-adic rational number, we agree to write a terminating expansion in the form (1) in which all digits from a certain point on are 0.

Thus every real number in I_0 is uniquely expressed by (1) and an infinite sequence of integers $\{a_n\}_{n=1,2}\cdots$ taking one of the values in R can be corresponded to a unique real number a in I_0 defined by

$$a=\sum_{n=1}^{\infty}\frac{a_n}{r^n}.$$

We call a real number ω to be simply normal to the base r if, for each j in R,

$$\lim_{N\to\infty}\frac{A_N(j;\omega)}{N}=\frac{1}{r}, \tag{2}$$