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AN ALTERNATIVE PROOF OF BIRMAN-HILDEN-VIRO'S THEOREM

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In this paper we shall give an alternative proof of the thorem of Birman-Hilden-Viro (the following theorem (i), (iii), (iv), c.f. [2], [3], [6]).

THEOREM. (i) *Every closed orientable 3-manifold M of Heegaard genus ≤ 2 is homeomorphic to the 2-fold branched covering space of S^3 with a 3-bridge link L as its branch line.*

(ii) *In particular, if M is a homology sphere, then L is a knot. (More generally L is a knot iff the order of the 1-dimensional homology group of M is odd.)*

(iii) *There is an algorithm to construct L from (a Heegaard diagram of) M.*

(iv) *M is homeomorphic to S^3 iff L is the trivial knot. (And hence there is an algorithm to decide whether M is S^3 or not.)*

(v) *Each equivalence class of Heegaard splittings determines a unique knot type.*

(vi) *L is not uniquely determined by M. (L depends on a Heegaard splitting of M.)*

REMARK 1. (iv) is proved as follows: If M is homeomorphic to S^3, then by (i) S^3 is the 2-fold branched covering of S^3 with L as its branch line. This gives an involution of S^3 with fixed points L. By the result of [8], L must be a trivial knot. Whether L is a trivial knot or not is decided by the algorithm of Haken in [9].

REMARK 2. (v) is proved in Theorem 8 in [3] and (vi) is proved in [5] and [10].

In the following we shall prove (i).

Suppose that M is a closed orientable 3-manifold of Heegaard genus ≤ 2. Then M has a Heegaard splitting of genus 2. Hence we may suppose that M is obtained by pasting suitably surfaces of two handle-bodies of genus 2. (See the figure 1.)

In this pasting let the loops d', e', f' on M_1 correspond to the loops d, e, f on M_2. In this case we may suppose that the loops a, b, c and the loops d', e', f' intersect transversally in only a finite number of points.

Moreover we may suppose that there is no section of the loops which bounds

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