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ALGEBRAIC INDEPENDENCE OF FIBONACCI RECIPROCAL SUMS ASSOCIATED WITH NEWTON'S METHOD

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1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0) \tag{1}$$

and $\{L_n\}_{n\geq 0}$ the sequence of Lucas numbers defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0). \tag{2}$$

There are many investigations on the arithmetic properties of reciprocal sums of products of Fibonacci or Lucas numbers. André-Jeannin [1] proved that the sums

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1}}$$

are expressed as explicit formulas, more precisely as linear combinations over $\mathbf{Q}(\sqrt{5})$ of the values of the Lambert series $\sum_{n=1}^{\infty} z^n/(1-z^n)$ at numbers of $\mathbf{Q}(\sqrt{5})$. It is well-known that

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1-\sqrt{5}}{2}.$$

(For the proof see (9) in the next section.) Brousseau [2] proved that

$$S_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+2}} = 2 - \sqrt{5}.$$

It is easily seen that

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