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ON THE GENERALIZED JOSEPHUS PROBLEM

Mar chuimhne air an *t*-ollamh Rob Alasdair Mac Fhraing nach mairean

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1. Introduction

The legendary problem of Josephus and the forty Jews and the problem of fifteen Christians and fifteen Tarks, and also some variants thereof, are widely well known (cf. [1], [6], [10], [11], [14]) and have been discussed and generalized mathematically by several authors (cf. e.g. [2], [3], [4], [5], [8]).

These problems, in a rather general form, may well be formulated thus: Let n and m be given positive integers; we arrange n distinct points, named $1, 2, \ldots, n$, in a circle in the natural order (the points adjacent to 1 being 2 and n if $n > 2$) and delete, starting from the point 1, every mth point in turn until all the points are removed. The problem is to determine the kth point $a_m(k,n)$ (sometimes called the kth Josephus number) to be deleted when n, m and k $(1 \leqq k \leqq n)$ are assigned. It is plain that

$$1 \leqq a_m(k,n) \leqq n$$

and

$$a_m(1,n) \equiv m \pmod{n}.$$

Consequently, if the validity is assumed of the congruence

$$(1) \qquad a_m(k+1, n+1) \equiv m + a_m(k,n) \pmod{n+1} \quad (1 \leqq k \leqq n),$$

then one can recursively determine all the numbers $a_m(k,n)$.

A simple proof of the congruence relation (1) which is due substantially to P. G. Tait [13], was given by R. A. Rankin [8] (see also [4]); however, it should be noted that the basic congruence (1) was practically known to Seki Takakazu (1642?–1708) in [12] and to Leonhard Euler (1707–1783) in [3] as well. On the basis of (1) Rankin [8] has established an algorithm for determining the last

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