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NEUTRAL HYPERKÄHLER STRUCTURES ON PRIMARY KODAIRA SURFACES

By

Hiroyuki KAMADA

1. Introduction

In the present paper, we study neutral hyperkähler structures on four-dimensional manifolds, which draw attention recently in differential geometry and especially in mathematical physics (cf. Hull [14], Ooguri-Vafa [20]). A neutral hyperkähler structure on a pseudo-Riemannian four-manifold M of metric signature (2, 2) consists of a neutral metric g and three endomorphisms $(I, 'J, 'K)$ on the tangent bundle TM of M such that

$$I^2 = -\mathrm{Id}, \quad 'J^2 = 'K^2 = \mathrm{Id} \quad \text{and} \quad I'J = -'JI = 'K; \tag{1}$$

$$g(V_1, V_2) = g(IV_1, IV_2) = -g('JV_1, 'JV_2) = -g('KV_1, 'KV_2) \tag{2}$$

for arbitrary vector fields V_1, V_2 on M, and that these structures enjoy some desired properties similar to the Kähler condition. We shall call a triple $(I, 'J, 'K)$ satisfying (1) a split-quaternion structure (or a paraquaternionic structure in some literature (cf. Blažić [4], García-Río et al. [10])), g satisfying (2) a compatible metric with $(I, 'J, 'K)$, and $(g, I, 'J, 'K)$ a neutral almost hyperhermitian structure. For a four-manifold M endowed with such a structure $(g, I, 'J, 'K)$, the invariance of g by I and the skew-invariance by $'J$ and $'K$ allow us to define three nondegenerate 2-forms $\Omega_I, \Omega_{'J}, \Omega_{'K}$, called the fundamental forms, as follows:

$$\Omega_I(V_1, V_2) := g(IV_1, V_2), \Omega_{'J}(V_1, V_2) := g('JV_1, V_2), \Omega_{'K}(V_1, V_2) := g('KV_1, V_2),$$

where V_1, V_2 are vector fields on M.

DEFINITION. A neutral almost hyperhermitian four-manifold $(M, g, I, 'J, 'K)$

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