TSUKUBA J. MATH.
Vol. 23 No. 1 (1999), 55–89

A CERTAIN GRAPH OBTAINED FROM A SET OF SEVERAL POINTS ON A RIEMANN SURFACE

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Introduction

0-1. Let M be a compact Riemann surface of genus $g \geq 2$, and let $P_1, P_2, \ldots, P_n$ be distinct points on M. We define the Weierstrass gap set $G(P_1, P_2, \ldots, P_n)$ by

$$G(P_1, P_2, \ldots, P_n) := \{(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbf{N}_0 \times \cdots \times \mathbf{N}_0 \mid \nexists \text{ meromorphic function } f \text{ on } M \text{ whose pole divisor } (f)_\infty \text{ is } \gamma_1 P_1 + \gamma_2 P_2 + \cdots + \gamma_n P_n\},$$

where $\mathbf{N}_0$ is the set of non-negative integers.

When $n = 1$, $G(P_1)$ is the set of Weierstrass gaps at P_1. One of the essential differences between the case $n = 1$ and the case $n \geq 2$ is that the cardinarity $\# G(P_1)$ is the constant g but $\# G(P_1, \ldots, P_n)$ $(n \geq 2)$ depends on the choice of M and the set of points $\{P_1, \ldots, P_n\}$ on M.

Kim has given formulas for $\# G(P_1, P_2)$ and shown the following inequalities

$$\frac{(g^2 + 3g)}{2} \leq \# G(P_1, P_2) \leq \frac{(3g^2 + g)}{2}.$$

Moreover he has proved that the upper bound $(3g^2 + g)/2$ can be realized if and only if "M is hyperelliptic and $|2P_1| = |2P_2| = g_2^1$" ([3]). The lower bound $(g^2 + 3g)/2$ can be attained by taking general points P_1 and P_2 on arbitrary M. This is stated in [1] without proof, and has been proved by Homma ([2]). He also has translated Kim's formulas into other practical ones, and added several interesting remarks in the case where M is a curve defined over a field of characteristic $p \geq 0$ ([2]). Through their works it seems to be helpful to use a certain type of graph $D^{(n)}$ defined as follows.

Definition 0-2 (Riemann-Roch Graph). Fix positive integers g and n. Let $\mathbf{e}_i$ be the n-tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ (i.e., the i-th component of $\mathbf{e}_i$ is 1) in $\mathbf{N}_0^n$.

Received October 3, 1997
Revised October 26, 1998