TSUKUBA J. MATH.
Vol. 21 No. 1 (1997), 449–482

A theorem in the geometry of numbers

By

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Let M be the exterior of a knot in the 3-sphere S^3 (or more generally a compact 3-manifold with a torus as boundary) and let $M(p,r)$ be the closed 3-manifold obtained from M by (p,r)-Dehn surgery. (p, r are co-prime integers.) Roughly speaking, the number of non-trivial representations of the fundamental group of $M(p,r)$ to PSL $(2,\boldsymbol{C})$ is given by the formula

$$\sum_{i=1}^{n}|\alpha_i p-\beta_i r|-e$$

So, if this number is positive, then $M(p,r)$ is not simply-connected. So, the calculation of this number is useful for studying Poincaré conjecture.

In this paper we shall prove a theorem about the functions of the above form, purely in the geometry of numbers, independent of the topology of 3-manifolds. We use Minkowski's theorem in proving this theorem. Moreover we introduce the notion of C-system as a tool for proving the theorem. In future we wish to apply this theorem to the study of Poincaré conjecture.

Let $L'=\boldsymbol{Z}\times\boldsymbol{Z}-\{(0,0)\}$

THEOREM 1. *Let α_i,β_i $(i=1,\dots,m)$, γ_j,δ_j $(j=1,\dots,n)$, e, f be real numbers such that $\alpha_i\beta_k-\beta_i\alpha_k\neq 0$ $(i\neq k)$, $\gamma_j\delta_\ell-\delta_j\gamma_\ell\neq 0$ $(j\neq l)$, $e>0$, $f>0$. Suppose that, for all $(x,y)\in L'$,*

$$\sum_{i=1}^{m}|\alpha_i x-\beta_i y|\geq e \tag{1}$$

and

$$\sum_{j=1}^{n}|\gamma_j x-\delta_j y|\geq f. \tag{2}$$

Then,

Received September 4, 1995.
Revised January 20, 1997.