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# HAUSDORFF APPROXIMATIONS ON HADAMARD MANIFOLDS AND THEIR IDEAL BOUNDARIES

Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

By

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## §1. Introduction

The concept of ideal boundary of Hadamard manifolds was first introduced by Eberlein and O'Neill [3], and then their Tits metrics were defined by Gromov [2] in the following manner.

Let $M$ be a Hadamard manifold, that is, a simply connected complete Riemannian manifold of nonpositive curvature. In what follows, geodesics are always assumed to be parametrized by arc length. Two geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ are said to be *asymptotic* if the distance function $t \to d_M(\gamma_1(t), \gamma_2(t))$ is bounded from above for all $t \geq 0$. Then the *ideal boundary* $M(\infty)$ of $M$ is defined to be the set of all asymptotic classes of geodesic rays in $M$. For $z_1, z_2 \in M(\infty)$ and $p \in M$, let $\gamma_1, \gamma_2$ be rays from $p$ to $z_1, z_2$. The function $t \to d(\gamma_1(t), \gamma_2(t))/t$ is then monotone non-decreasing and is bounded from above by 2. Thus we can define a metric $l$ on $M(\infty)$ by

$$l(z_1, z_2) := \lim_{t \to \infty} \frac{d(\gamma_1(t), \gamma_2(t))}{t}.$$

It is easy to see that the definition of $l$ is independent of the choice of $p$ and that $l$ is indeed a metric on $M(\infty)$. The *Tits metric* $Td(\cdot,\cdot)$ is then the interior metric $l_i$ induced from this metric.

Subsequently, the concept of ideal boundary was also defined for other classes of Riemannian manifolds in a similar fashion. Among them, Kasue [5] defined it on asymptotically nonnegatively curved manifolds, and Shioya [8], [9] on complete open surfaces admitting total curvature.

On the other hand, we know the concepts of rough isometry and Hausdorff approximation between two metric spaces, which preserve certain asymptotic properties, in the following way (cf. Kanai [4]).

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