TSUKUBA J. MATH.
Vol. 20 No. 2 (1996), 399–411

A NOTE ON FREE DIFFERENTIAL GRADED ALGEBRA RESOLUTIONS

By

D. TAMBARA

Introduction

We work ove a field k. A differential graded algebra (dga for short) in this paper is a graded k-algebra $U=\oplus_{n\geq 0}U_n$ with differential d of degree -1. Given a k-algebra R, it is well-known that there exists a free dga resolution $\varepsilon: U\rightarrow R$ (Baues [2]). That is, U is a dga which is free as a graded algebra, ε is a dga map, and the sequence

$$\cdots\xrightarrow{d}U_n\xrightarrow{d}\cdots\xrightarrow{d}U_0\xrightarrow{\varepsilon}R\rightarrow 0$$

is exact. Such a resolution is thought of as a prolongation of a presentation of R by generators and relations, and expected to contain lots of information about homology of R. Although free dga's frequently appear in homotopical algebra such as [2], not much seems to be known about the structure of free dga resolutions of algebras.

We study here a relationship between a free dga resolution of R and a free bimodule resolution of the R-bimodule R. Let U be a dga which is free on a graded space E, and $\varepsilon: U\rightarrow R$ an augmentation map. We construct a complex $R\otimes E\otimes R$ of free R-bimodules with augmentation $\sigma: R\otimes E\otimes R\rightarrow\Omega_R$, where Ω_R is the kernel of the multiplication map $R\otimes R\rightarrow R$. If ε is a resolution, then so is σ (Proposition 1.2). The converse is true when R is a connected graded algebra and U,ε are taken to be compatible with the grading of R (Theorem 3). Therefore, the verification of the exactness of $\varepsilon: U\rightarrow R$ reduces to that of $\sigma: R\otimes E\otimes R\rightarrow\Omega_R$, which is much easier.

Using this criterion, we give explicit free dga resolutions of Koszul algebras and their generalizations.

NOTATION. For a graded module $M=\oplus_{n\geq 0}M_n$, we write $M_+=\oplus_{n>0}M_n$. For a

Received August 31, 1994.
Revised May 31, 1995.