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NON-TRIVIALITY OF CERTAIN FINITELY-PRESENTED GROUPS

By

Moto-o TAKAHASHI

In this paper we shall prove the following theorem.

THEOREM. *Let p, q, r, s be integers greater than 2. Then, the group*

$$\langle a, b \mid a^{p}=b^{q}=(ab)^{r}=(a^{-1}b)^{s}=1\rangle$$

is non-abelian and hence non-trivial.

REMARK 1. The groups of this type were studied in [1], [2], [3]. But the above general theorem was not established.

REMARK 2. If one of p, q, r, s is 2 in the above group presentation, then there are many cases when the group becomes trivial.

PROOF OF THE THEOREM. We define matrices A, $B\in \mathrm{SL}(3, \boldsymbol{C})$ such that A and B do not commute and that $A^{p}=B^{q}=(AB)^{r}=(A^{-1}B)^{s}=E$.

Let ω_{p} be a primitive p-th root of 1, ω_{q} be a primitive q-th root of 1, ω_{r} be a primitive r-th root of 1, and ω_{s} be a primitive s-th root of 1. Since p, q, r, $s>2$, we have $\omega_{p}\neq\omega_{p}^{-1}$, $\omega_{q}\neq\omega_{q}^{-1}$, $\omega_{r}\neq\omega_{r}^{-1}$, $\omega_{s}\neq\omega_{s}^{-1}$.

Now let

$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega_{p} & 0\\ 0 & 0 & \omega_{p}^{-1}\end{pmatrix},\quad B=\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix},$$

where b_{ij}'s will be determined later. Obviously, $A^{p}=E$.

In order that $B^{q}=E$, it is sufficient that the characteristic polynomial $\chi_{B}(t)$ of B is $(t-1)(t-\omega_{q})(t-\omega_{q}^{-1})$, for it is a factor of $t^{q}-1$ so $t^{q}-1=f(t)\chi_{B}(t)$, for some polynomial $f(t)$ and hence $B^{q}-E=f(B)\chi_{B}(B)=0$ (the zero matrix).

From

$$\begin{aligned}\chi_{B}(t)&=t^{3}-(b_{11}+b_{22}+b_{33})t^{2}\\ &\quad+\left\{\begin{vmatrix}b_{22} & b_{23}\\ b_{32} & b_{33}\end{vmatrix}+\begin{vmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{vmatrix}+\begin{vmatrix}b_{11} & b_{13}\\ b_{31} & b_{33}\end{vmatrix}\right\}t-\begin{vmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{vmatrix}\\ &=t^{3}-(1+\omega_{q}+\omega_{q}^{-1})t^{2}+(1+\omega_{q}+\omega_{q}^{-1})t-1,\end{aligned}$$

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