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ON THE COMPLETE RELATIVE HOMOLOGY AND COHOMOLOGY OF FROBENIUS EXTENSIONS

By

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Introduction.

Let G be a finite group, K a subgroup of G and M a left G-module. Then for $r\in\boldsymbol{Z}$ the complete relative homology group $H_r(G, K, M)$ and cohomology group $H^r(G, K, M)$ are defined in [6]. Let 1 be the unit element of G. For the case of $K=\{1\}$ $H_r(G, K, M)\cong H^{-r-1}(G, K, M)$ holds. But it is not true that for any G, K, M and r there exists an isomorphism from $H_r(G, K, M)$ into $H^{-r-1}(G, K, M)$. In fact, in [6, p. 262] there are G, K and M such that $H_r(G, K, M)\cong\boldsymbol{Z}/2\boldsymbol{Z}$ and $H^r(G, K, M)=0$ for all $r\in\boldsymbol{Z}$. And if we set $M=\boldsymbol{Q}/\boldsymbol{Z}$ in [6, p. 262], $H_r(G, K, M)=0$ and $H^r(G, K, M)\cong\boldsymbol{Z}/2\boldsymbol{Z}$ hold for all $r\in\boldsymbol{Z}$.

Let Λ be an algebra over a commutative ring K and Γ a subalgebra such that the ring extension Λ/Γ is a Frobenius extension. In section 1 we shall introduce the complete relative cohomology group $H^r(\Lambda, \Gamma, -)$ and homology group $H_r(\Lambda, \Gamma, -)$ for $r\in\boldsymbol{Z}$. When the ring extension Γ/K is also a Frobenius extension, We can define a K-homomorphism $\Psi^r_{\Lambda/\Gamma}: H_r(\Lambda, \Gamma, (-)^{\Delta})\rightarrow H^{-r-1}(\Lambda, \Gamma, -)$ for $r\in\boldsymbol{Z}$, where Δ is the Nakayama automorphism. The main purpose of this paper is to show necessary and sufficient conditions on which $\Psi^r_{\Lambda/\Gamma}$ is an isomorphism. Theorems 6.3, 7.1 and 7.2 provide the necessary and sufficient conditions. In section 8 we apply our results to extensions defined by a finite group G and a subgroup K. In generalization of the well-known duality for Tate cohomology we show that $H_r(G, K, -)\cong H^{-r-1}(G, K, -)$ if and only if K is a Hall subgroup of G.

1. Complete relative homology.

Throughout this paper, let Λ be an algebra over a commutative ring K and Γ a subalgebra such that the ring extension Λ/Γ is a (projective) Frobenius extension in the sense of [9]. Since Λ/Γ is a Frobenius extension, there exist elements $R_1, \cdots, R_n, L_1, \cdots, L_n$ in Λ and a $\Gamma-\Gamma$-homomorphism $H\in\mathrm{Hom}({}_{\Gamma}\Lambda_{\Gamma},$

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