TSUKUBA J. MATH.
Vol. 17 No. 2 (1993), 513–535

AVERAGE ORDER OF THE DIVISOR FUNCTIONS WITH NEGATIVE POWER WEIGHT

Dedicated to Professor Katsumi Shiratani on his 60th birthday

By

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1. Introduction.

In this paper we are primarily concerned with the study of the sums of the sum-of-divisors function $\sigma_a(n)$ with negative power weight $n^{-t}(t>0)$, i.e. the sums of the form

$$\sum_{n\leq x} n^{-t}\sigma_a(n)$$

and we also study the averages of associated error terms. Throughout the paper, we shall refer to [6] as I and whose results we cite e.g. as I-Theorem 1. First we consider the case $0\leqq a-t\in \boldsymbol{Z}$, where $\boldsymbol{Z}$ denotes the set of all rational integers, and prove Theorem 1 which generalizes and in some cases corrects MacLeod's Theorem 8[8]. This case is easier to handle although the needed calculations are rather long. And the special case $a=t$ of this is the starting point of the investigation of the case $a<t$. In this case our approach, which depends on MacLeod's back-track method (Lemma 1 below), is not so effective for a large, and we have to restrict ourselves to the narrower range $0\leqq a\leqq 3$ which, however, covers and interpolates all the formulas obtained by MacLeod. In the case of general t we appeal to induction, and in order to guess the forms of the formulas, we have to calculate out all the cases $t=a+1$, $t=a+2$, $t=a+3$, the last being the initial value of t for induction. Here we take the instructive standpoint and calculated out all these three cases successively and then give the form for $t\geqq a+3$, since each independent formula seems to have its own interest. Except for integral values of a, our interpolating formulas involve various negative powers of x with extremely complicated and clumsy coefficients, but in some cases they are absorbed in the error terms by just multiplying the log-factor. The main reasons why we restrict ourselves to $0\leqq a\leqq 3$ are the complication of these coefficients as well as inapplicability of Lemma 8. However, we state the formulas for $a>3$ as well, only for $t=$

Received April 15, 1992, Revised February 10, 1993.