TSUKUBA J. MATH.
Vol. 17 No. 1 (1993), 143—158

HALF CONFORMALLY FLAT STRUCTURES AND THE DEFORMATION OBSTRUCTION SPACE

Dedicated to Professor H. Nakagawa on his sixtieth birthday

By

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1. A compact connected oriented Riemannian 4-manifold (M, g) is called half conformally flat, or a Riemannian metric g on M is called self-dual or anti-self-dual, when $W^{-}=0$ or $W^{+}=0$ where $W^{\pm}$ is the self-dual (anti-self-dual) part of the Weyl conformal curvature tensor W of g.

We denote for an arbitrary Riemannian metric g by $R=(R_{ijkl})$, $Ric=(R_{ij})$ and ρ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. Then the Weyl conformal curvature tensor $W=(W_{ijkl})$, considered as a section of the symmetric product bundle $S^{2}(\Omega^{2})$, is defined by

$$R=W+L\owedge g$$

($L=1/2\ (Ric-(\rho/6)g)$ is the Schouten tensor and $\owedge$ is the Kulkarni-Nomizu product).

In terms of the Hodge star operator the bundles Ω^{2} and $S^{2}(\Omega^{2})$ decompose as $\Omega^{2}=\Omega^{+}\oplus\Omega^{-}$ and $S^{2}(\Omega^{2})=S^{2}(\Omega^{+})\oplus(\Omega^{+}\otimes\Omega^{-})\oplus(\Omega^{-}\otimes\Omega^{+})\oplus S^{2}(\Omega^{-})$, respectively and then the tensors R and W split as $R=\begin{pmatrix} R^{+} & R^{+-} \\ R^{-+} & R^{-} \end{pmatrix}$ and $W=\begin{pmatrix} W^{+} & 0 \\ 0 & W^{-} \end{pmatrix}$ in such a way that $R^{\pm}=W^{\pm}+(\rho/12)I$.

The notion "half conformal flatness" depends only on a conformal structure $[g]$, the conformal equivalence class represented by a Riemannian metric g, because W and the Hodge star operator are conformal invariants.

The significance of the half conformally flat structure is that it ensures the integrability of the almost complex structure which is naturally defined on the twistor space $Z_{M}\rightarrow M$, the unit sphere bundle of Ω^{+} such that Z_{M} becomes a complex 3-fold admitting a real structure ([1]).

Like Yang-Mills instantons on 4-manifolds, every half conformally flat structure $[g]$ enjoys an elliptic complex at any representative within $[g]$ provided $W=W^{-}$ i.e., $W^{+}=0$

Receied November 8, 1991, Revised May 27, 1992.