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HYPOELLIPTICITY FOR A CLASS OF DEGENERATE ELLIPTIC OPERATORS OF SECOND ORDER

Dedicated to Professor Mutsuhide MATSUMURA on his sixtieth birthday

By

Michiharu SUZUKI

§1. Introduction and results.

Fedii [1] studied hypoellipticity for operators of the form $L=D_1^2+\phi(x_1)^2D_2^2$ in $\boldsymbol{R}^2$, and proved that L is hypoelliptic in $\boldsymbol{R}^2$ if $\phi(x_1)\in C^\infty(\boldsymbol{R})$ and $\phi(x_1)>0$ for $x_1\neq 0$. Hörmander's results in [2] can not be applicable to L when $\phi(x_1)$ has a zero of infinite order. Compared with higher dimensional cases, the problem in $\boldsymbol{R}^2$ becomes much simpler. So one can expect that one investigates hypoellipticity for more general operators in $\boldsymbol{R}^2$. In this paper we shall give sufficient conditions of hypoellipticity for operators of the form $P(x, D)=D_1^2+\alpha(x)D_2^2+\beta(x, D)$ in $\boldsymbol{R}^2$, where $x=(x_1, x_2)\in\boldsymbol{R}^2$, $\alpha(x)\in C^\infty(\boldsymbol{R}^2)$ is non-negative and $\beta(x, D)$ is a properly supported classical pseudodifferential operator of order 1. In doing so, we need general criteria for hypoellipticity, which are improvements of ones obtained by Morimoto [5] (see Theorem 1.1 below).

Let us define the usual symbol classes $S_{1,0}^{m,\mathrm{loc}}$ and $S_{1,0}^m$. We say that a symbol $p(x, \xi)$ belongs to $S_{1,0}^{m,\mathrm{loc}}$ (resp. $S_{1,0}^m$) if $p(x, \xi)\in C^\infty(T^*\boldsymbol{R}^n)$ and if for any compact subset K of $\boldsymbol{R}^n$ and for any multi-indices α and β (resp. for any multi-indices α and β there is $C_{\alpha,\beta}\equiv C_{K,\alpha,\beta}>0$ (resp. $C_{\alpha,\beta}>0$) such that $|p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$ for $x\in K$ and $\xi\in\boldsymbol{R}^n$ (resp. for $(x, \xi)\in T^*\boldsymbol{R}^n$), where $m\in\boldsymbol{R}$, $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$, $D_x=-i\partial_x$, $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $T^*\boldsymbol{R}^n$ is identified with $\boldsymbol{R}^n\times\boldsymbol{R}^n$. We denote by $L_{1,0}^m$ the set of the pseudodifferential operators whose symbols belong to $S_{1,0}^{m,\mathrm{loc}}$. Let $P(x, D)\in L_{1,0}^m$ be a properly supported pseudodifferential operator, and let $z^0=(x^0, \xi^0)\in T^*\boldsymbol{R}^n\setminus 0$ $(\cong\boldsymbol{R}^n\times(\boldsymbol{R}^n\setminus\{0\}))$. It is said that $P(x, D)$ is microhypoelliptic at z^0 if there is a conic neighbourhood $\mathcal{V}$ of z^0 in $T^*\boldsymbol{R}^n\setminus 0$ such that $WF(u)\cap\mathcal{V}=WF(Pu)\cap\mathcal{V}$ if $u\in\mathcal{D}'(\boldsymbol{R}^n)$. We also say that $P(x, D)$ is microhypoelliptic in a conic in a conic set $\mathcal{V}(\subset T^*\boldsymbol{R}^n\setminus 0)$ (resp. in $\Omega(\subset\boldsymbol{R}^n)$) if $P(x, D)$ is microhypoelliptic at each $(x, \xi)\in\mathcal{W}$ (resp. at

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