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IDEALS ON ω WHICH ARE OBTAINED FROM HAUSDORFF-GAPS

By

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Let $\mathcal{G}$ be a Hausdorff gap in ${}^{\omega}\omega$. Hart and Mill [2] defined the ideal $I_{\mathcal{G}}$ which is the family of all subsets of ω whose restriction of $\mathcal{G}$ is filled. In this paper, we shall show two results (Theorems 1, 6) about these ideals.

Our notions and terminology follow the usual use in set theory. Let X be a subset of ω and f, g functions from X to ω. g dominates f (denoted by $f \prec g$), if $\{n \in X; g(n) \leqq f(n)\}$ is finite. Let κ and λ be infinite cardinals. A pair of sequence $\langle\langle f_{\alpha} | \alpha < \kappa\rangle | \langle g_{\beta} | \beta < \lambda\rangle\rangle$ is called a (κ, λ)-gap, if the following (1), (2) are satisfied.

(1) $f_{\alpha}, g_{\beta}: \omega \to \omega$, for any $\alpha < \kappa$, $\beta < \lambda$.

(2) $f_{\alpha} \prec f_{\gamma} \prec g_{\delta} \prec g_{\beta}$, for any $\alpha < \gamma < \kappa$, $\beta < \delta < \lambda$.

A (κ, λ)-gap $\langle\langle f_{\alpha} | \alpha < \kappa\rangle | \langle g_{\beta} | \beta < \lambda\rangle\rangle$ is unfilled, if there does not exist a function $h: \omega \to \omega$ such that, for all $\alpha < \kappa$, $\beta < \lambda$, $f_{\alpha} \prec h \prec g_{\beta}$. We call an unfilled (ω_1, ω_1)-gap a Hausdorff gap (H-gap). The following fact is well-known.

FACT. For any regular cardinals κ and λ with $(\kappa, \lambda) \neq (\omega_1, \omega_1)$, there exists a generic extension W such that W preserves all cardinals and, in W, there are no unfilled (κ, λ)-gap.

In contrast to this fact, the following theorem holds about H-gaps.

THEOREM (Hausdorff [1, Theorem 4.3]). *There is an H-gap.*

Let $\mathcal{G} = \langle\langle f_{\alpha} | \alpha < \omega_1\rangle | \langle g_{\alpha} | \alpha < \omega_1\rangle\rangle$ be a (ω_1, ω_1)-gap. Following [2], we define the ideal $I_{\mathcal{G}}$ by

$$I_{\mathcal{G}} = \{x \subset \omega; \exists h: x \to \omega \forall \alpha < \omega_1 (f_{\alpha} \upharpoonright x \prec h \prec g_{\alpha} \upharpoonright x)\}.$$

It is easy to see that

$$\omega \in I_{\mathcal{G}} \text{ if and only if } \mathcal{G} \text{ is filled,}$$

$$\mathrm{Fin} = \{x \subset \omega; x \text{ is finite}\} \subset I_{\mathcal{G}}.$$

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