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APPROXIMATING THE HOMOTOPY SEQUENCE OF A PAIR OF SPACES

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0. Introduction.

In his book [7] H. Toda computed the homotopy groups $\pi_{n+k}(S^n)$ for $k\leqq 19$. Although he permitted himself to use any methods and insights that were available to him at the time, his principal technique was to exploit his "composition method". Very briefly the composition method is an inductive procedure by which composition classes vanishing in a particular "stem" k give rise to secondary and higher order composition classes (Toda brackets) in stem $k+1$ that are "detected by" (i. e. shown to have a non-zero image under) a Hopf invariant homomorphism, after which their orders and relations to other elements are determined. This paper is intended as a contribution toward an analysis of the composition method and its development into a more generally applicable technique for computing homotopy groups.

Let $i: A\rightarrow X$ be an inclusion map and let us suppose that the homotopy groups of X are less well known than those of A. (In Toda's composition method $i: S^n\rightarrow \Omega S^{n+1}$ is the suspension inclusion). Then we have available the relative homotopy sequence

$$(0.1)\qquad \cdots\longrightarrow \pi_n(A)\longrightarrow \pi_n(X)\longrightarrow \pi_n(X, A)\longrightarrow \pi_{n-1}(A)\longrightarrow\cdots.$$

In general $\pi_n(X, A)$ will not be known but we may be able to approximate it via a map $h: X\rightarrow B$ with $h(A)=*$ (the base point) and where B is a space whose homotopy groups are (better) known. Then we can regard

$$(0.2)\qquad \cdots\longrightarrow \pi_n(A)\xrightarrow{i_*}\pi_n(X)\xrightarrow{H}\pi_n(B)\xrightarrow{\Delta}\pi_{n-1}(A)\longrightarrow\cdots$$

(where $H=h_*$) as an approximation to 0.1. Of course if $h_*: \pi_n(X, A)\rightarrow\pi_n(B)$ is not an isomorphism then 0.2 may not be exact. Indeed the operator Δ may not be well-defined. However a partial function (defined on the kernel of i_*) $\Delta^{\leftarrow}: \pi_{n-1}(A)\rightarrow\pi_n(B)$ can be defined (with a degree of indeterminacy) using Toda brackets. If $\mu\in\pi_{n-1}(A)$ is such that $i_*\mu=0$, let

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