A NOTE ON THE NAKAYAMA CONJECTURES

By

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The *Nakayama conjecture* states that every finite dimensional algebra of infinite dominant dimension is QF (see [13], [14], [16]). The *generalized Nakayama conjecture* (*GNC*) of Auslander and Reiten [2], whose verification would imply Nakayama's conjecture, maintains that the injective envelopes of all the simple modules over an artin algebra Λ appear as direct summands of terms in the minimal injective resolution of Λ; or equivalently, if $_\Lambda S$ is simple then there is an $i \geq 0$ such that $\mathrm{Ext}_\Lambda^i(S, \Lambda) \neq 0$. Jans [12] proved that if R is a left noetherian ring such that the right finitistic global dimension of R, $\sup\{\mathrm{Pd}(N_R) \mid \mathrm{Pd}(N_R) < \infty\}$ (taken over finitely generated right R-modules), is finite then if $_R M \neq 0$ is finitely generated there is an $i \geq 0$ with $\mathrm{Ext}_R^i(M, R) \neq 0$. The *finitistic dimension conjecture* (arising from questions raised in [3] and [12]), asserts that the finitistic global dimension of every finite dimensional algebra is finite. These conjectures lead us to consider the intermediate condition on finitely generated modules

$$\mathrm{EXT}_R(M, R) = 0 \quad \text{implies} \quad M = 0$$

where $\mathrm{EXT}_R(M, R) = \oplus_{i \geq 0} \mathrm{Ext}_R^i(M, R)$. We shall say that a ring for which this condition holds satisfies *the strong Nakayama conjecture* (*SNC*).

In this note we present a generalization (Theorem 2) of a theorem of Iwanaga [11] by verifying the SNC for a class of rings that properly includes the noetherian rings for which the regular module R_R has finite injective dimension. We introduce a condition that appears to be considerably weaker than the ultimately closed condition of [12], [13] and [16]; and we also show (Proposition 4) that it yields the SNC for a class of algebras that contains those for which the Nakayama conjecture was verified by Mueller in [13] and is closely related to those for which the GNC was verified by Auslander and Reiten in [2].

The finitistic dimension conjecture has recently been verified for monomial (or zero-relation) algebras by Green, Kirkman, and Kuzmanovich [9] and by