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ON A MEAN-VALUE THEOREM CONCERNING DIFFERENCES OF TWO K-TH POWERS

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1. Introduction. For positive integers k, r, let $t_k(r)$ denote the number of pairs $(m, n)\in \boldsymbol{N}\times \boldsymbol{Z}$ with $m^k-|n|^k=r$. To study the average order of $t_k(r)$, one considers the summatory function $T_k(x)=\sum_{1\leq r\leq x} t_k(r)$ (x a large real variable). It has been proved by E. Krätzel [3] that, for $k\geqq 3$ and some small $\varepsilon_0>0$,

$$T_k(x)=c_1(k)x^{2/k}+c_2(k)x^{1/(k-1)}+c_3(k)F_k(x)x^{1/k-1/k^2}+O(x^{2/(3k)-\varepsilon_0}), \qquad (1)$$

where

$$c_1(k)=\Gamma^2\left(\frac{1}{k}\right)\left(2k\cos\left(\frac{\pi}{k}\right)\Gamma\left(\frac{2}{k}\right)\right)^{-1}, \qquad c_2(k)=2\zeta\left(\frac{1}{k-1}\right)k^{-1/(k-1)},$$

$$c_3(k)=\pi^{-1-1/k}\left(\frac{k}{2}\right)^{1/k-1}\Gamma\left(\frac{1}{k}\right), \qquad F_k(x)=\sum_{n=1}^{\infty} n^{-1-1/k}\sin\left(2\pi n x^{1/k}+\frac{\pi}{2k}\right),$$

hence $F_k(x)=O(1)$ and $F_k(x)=\Omega_{\pm}(1)$ as $x\to\infty$. For $k=2$, the problem is essentially equivalent to the Dirichlet divisor problem, since (cf. e.g. [4])

$$T_2(x)=D(x)-2D\left(\frac{x}{2}\right)+2D\left(\frac{x}{4}\right), \qquad D(x):=\sum_{0<mn\leq x} 0.$$

2. Statement of result. In this note, we apply the modern technique for the estimation of exponential sums (the "discrete Hardy-Littlewood method", due to Bombieri, Iwaniec, Mozzochi, Huxley and Watt), together with a refined analysis of the special functions involved, in order to improve the error term in the above estimate.

THEOREM. *For any real number* $k\geqq 2$, let $T_k(x)$ *denote the number of lattice points* $(m, n)\in \boldsymbol{N}\times \boldsymbol{Z}$ *with* $0<m^k-|n|^k\leqq x$. *If* $k\geqq 38/13$, *we have the asymptotic*

$$T_k(x)=c_1(k)x^{2/k}+c_2(k)x^{1/(k-1)}+c_3(k)F_k(x)x^{1/k-1/k^2}+\Delta_k(x)$$

with

$$\Delta_k(x)=O(x^{25/(38k)+\varepsilon}) \qquad \textit{for any} \quad \varepsilon>0.$$

Consequently, for $k>38/13$,

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