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SATURATED SETS FOR GENERALIZED CARTAN MATRICES

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

By

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0. Introduction.

In the theory of Kac-Moody Lie algebras, it is important to know the set of imaginary roots and the set of weights for integrable modules. We will study these two kinds of sets using the idea of saturated sets (cf. [1]), and show the following theorems.

THEOREM 1 ([3], [4], [8]). *Let A be a generalized Cartan matrix, and $\mathfrak{g}$ the Kac-Moody Lie algebra of type A. Then the root system Δ, with simple roots $\Pi=\{\alpha_1, \cdots, \alpha_n\}$, of $\mathfrak{g}$ is uniquely characterized by the following properties:*

(1) *Δ is a saturated set,*

(2) *$\Delta=-\Delta$,*

(3) *$k\alpha_i\in\Delta \Leftrightarrow k=0, \pm 1$ for all $\alpha_i\in\Pi$ and $k\in\boldsymbol{Z}$,*

(4) *$\beta<_{\Pi}0$ or $0<_{\Pi}\beta$ for each $\beta\in\Delta$,*

(5) *if $\beta\in\Delta$ and $ht(\beta)>1$, then there exists some $\alpha_i\in\Pi$ such that $\beta-\alpha_i\in\Delta$.*

A generalized Cartan matrix will be simply called a GCM.

THEOREM 2. *Let V be a standard $\mathfrak{g}$-module. Then the set Λ of weights for V is uniquely characterized by the following properties:*

(1) *Λ is a saturated set,*

(2) *there exists $\lambda\in\Lambda$ such that $\mu<_{\Pi}\lambda$ for all $\mu\in\Lambda$,*

(3) *if $ht(\lambda-\mu)>0$ for $\mu\in\Lambda$, then there exists some $\alpha_i\in\Pi$ such that $\mu+\alpha_i\in\Lambda$.*

Such a subset Λ is sometimes denoted $\Lambda(\lambda)$. Let $L=\bigoplus_{i=1}^{n}\boldsymbol{Z}\alpha_i$ and $L_-=\{\alpha\in L\mid \alpha<_{\Pi}0, \alpha\neq 0\}$. A nonzero element $\alpha\in L$ is called connected if $\mathrm{Supp}(\alpha)$ is connected. Let R be the subset of L consisting of all the connected elements. Put $R_-=R\cap L_-$ and $C=R_-\cup\{0\}\cup(-R_-)$. Let Δ_-^{im} be the set of negative imaginary roots of $\mathfrak{g}$.

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