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LOGARITHMIC UNIFORM DISTRIBUTION OF $(\alpha n+\beta\log n)$

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1. Introduction

A sequence $\omega=(x_n)_{n=1}^{\infty}$ of real numbers is said to be uniformly distributed modulo 1 if the proportion of indices $n\leq N$ such that the fractional parts $\{x_n\}$ are contained in an interval $I\subseteq[0,1)$ is asymptotically equal to the length of I. Put $\chi(x;y)=1$ for $\{y\}<x$ and $\chi(x;y)=0$ otherwise; then ω is uniformly distributed if and only if

$$(1)\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi(x;x_n)=x \quad \text{for} \quad 0<x<1.$$

It is well known (cf. the monographs [1] and [3]) that (1) is equivalent to

$$\lim_{N\to\infty}D_N^*(\omega)=0,$$

where

$$D_N^*(\omega)=\sup_{0\leq x<1}\left|\frac{1}{N}\sum_{n=1}^{N}\chi(x;x_n)-x\right|$$

denotes the discrepancy of the sequence ω. The systematic study of uniformly distributed sequences was initiated by H. Weyl [9]. Well known examples of uniformly distributed sequences are (αn) with irrational α and $(\sqrt{n})$; $(\log n)$ is known not to be uniformly distributed, but Tsuji [8] proved that

$$(2)\qquad \lim_{N\to\infty}\frac{1}{\sum_{n=1}^{N}\frac{1}{n}}\sum_{n=1}^{N}\frac{1}{n}\chi(x;x_n)=x \quad (0<x<1)$$

for $x_n=\log n$. A sequence $\omega=(x_n)$ with this property is said to be uniformly distributed with respect to the logarithmic mean. This is equivalent to

$$\lim_{N\to\infty}D_N(\omega)=0,$$

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