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H-SEPARABILITY OF GROUP RINGS
(In memory of Professor Akira Hattori)

By

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Let $k[G]$ be the group ring of a finite group G with a coefficient field k. Assume that the characteristic of k does not divide the order of G. Let H be a subgroup of G, Δ the centralizer of $k[H]$ in $k[G]$ and D the double centralizer of $k[H]$ in $k[G]$. The purpose of this paper is to prove that $k[G]$ is an H-separable extension of D. For this, a unit in the center C of $k[G]$ plays a fundamental role (Lemma 1). Besides, we can prove the well known facts that $k[G]$ is (finitely generated) projective over C and $k[G]$ is a central separable algebra over C, explicitely, by use of this unit.

Denote by g_x and c_x the number and the sum of elements in the conjugate class of G containing the element x of G, respectively.

LEMMA 1. *$u=\Sigma_{c_x}(1/g_x)c_xc_{x^{-1}}$ is a unit in C.*

PROOF. We first prove that $\{(1/g_x)c_x\}$ and $\{c_{x^{-1}}\}$ form a dual base of C over k. Let $c_yc_x=\Sigma_{c_z}c_za_{zx}$ where a_{zx} are integers. This means that each z_k $(1\leqq k\leqq g_z)$ conjugated to z, appears in c_yc_x a_{zx} times, that is, for fixed k, the number of pairs (i, j) such that $y_ix_j=z_k(1\leqq i\leqq g_y, 1\leqq j\leqq g_x)$ is equal to a_{zx}. So, the number of terms $x_j^{-1}=z_k^{-1}y_i(1\leqq j\leqq g_x)$ is $a_{zx}g_z$ in $c_{z^{-1}}c_y$ and $c_{z^{-1}}c_y=\cdots+(a_{zx}g_z/g_x)c_{x^{-1}}+\cdots$. This proves that $((1/g_z)c_{z^{-1}})c_y=\Sigma c_{x^{-1}}a_{zx}$ $((1/g_x)c_{x^{-1}})$ or equivalently $\{(1/g_x)c_x\}$ and $\{c_{x^{-1}}\}$ form a dual base of C over k. Now C is a separable k-algebra in the sense of that, for any field extension L of k, C_L is a semisimple L-algebra. Then $u=\Sigma_{c_x}(1/g_x)c_xc_{x^{-1}}$ is a unit in C by Theorem 71. 6 in [2] p.482.

Let v be the inverse of u in C, $uv=1$.

COROLLARY 2. *$\Sigma_{c_x}(1/g_x)c_x\otimes c_{x^{-1}}v$ is a separability idempotent in $C\otimes_kC$.*

PROOF. It is clear that $c(\Sigma(1/g_x)c_x\otimes c_{x^{-1}}v)=(\Sigma(1/g_x)c_x\otimes c_{x^{-1}}v)c$ for any $c\in C$ and $\Sigma(1/g_x)c_xc_{x^{-1}}v=1$.

Let p be the map of $k[G]$ to C defined by $p(a)=(1/n)\Sigma_{x\in G}xax^{-1}$ for $a\in k[G]$, where n is the order of G. The map p is the projection of $k[G]$ to C. Then p is an element of $\mathrm{Hom}_C(k[G], C)$ which has a left $k[G]$-module structure in the usual way.

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