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A SEQUENCE ASSOCIATED WITH THE ZEROS OF THE RIEMANN ZETA FUNCTION

By

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Let $\zeta(s)$ be the Riemann zeta function, let C_n denote the generalized Euler constants associated with $\zeta(s)$, i. e.,

$$C_n=\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\log^n k/k-\log^{n+1}N/(n+1)\right\},$$

and define the numbers $\delta_n(n\geqslant 1)$ by

$$\delta_1=1+\frac{1}{2}C_0-\frac{1}{2}\log\pi-\log 2,$$

and, for $n\geqslant 2$,

(1)
$$\delta_n=1-(1-2^{-n})\zeta(n)+n\sum_{h=1}^{n}\frac{1}{h}\sum_{\substack{j_1+\cdots+j_h=n-h\\ j_1\geqslant 0,\cdots,j_h\geqslant 0}}\prod_{b=1}^{h}\frac{C_{j_b}}{j_b!}.$$

Then we have, for $n\geqslant 1$,

(2)
$$\delta_n=\sum_{\rho}\rho^{-n},$$

where the sum Σ_ρ is taken over all complex zeros ρ of $\zeta(s)$(see [2]).

In this paper we shall study the sequence $\{\delta_n\}$ and prove four theorems. In Theorem 1 we shall give an expression of δ_n, and, using it, we shall derive, in Theorem 2, a necessary and sufficient condition for the truth of the Riemann hypothesis. The expression of δ_n will be specified in Theorem 3 under the Riemann hypothesis, and in the final Theorem 4, we shall give an upper bound for the quantity $|\delta_n|$.

We start with the following lemma.

LEMMA. *Let m be a positive integer, let $a_j(j=1,2,\cdots,m)$ be real numbers, and let $b_j(j=1,2,\cdots,m)$ be mutually distinct positive numbers $<\pi$. If*

$$\sum_{j=1}^{m}a_j\cos b_j n\longrightarrow 0,$$

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