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A REMARK ON HIGMAN'S RESULT ABOUT SEPARABLE ALGEBRAS

In memory of Professor Akira HATTORI

By

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1. Introduction.

Let k be a field with arbitarary characteristics and A a finite dimentional k-algebra. Then A is said to be a separable algebra if for any extension field E of k, $A\otimes_k E$ is a semisimple E-algebra. Let $a_1, \cdots, a_n$ be a k-basis of A and $T_{A|k}=T$ a trace form of A over k, that is, for $x\in A$, if $a_i x=\sum_j \alpha_{ij}a_j$ $(\alpha_{ij}\in k)$ then $T(x)=\Sigma\alpha_{ii}$.

It is well known that if A is a finite extension field of k, then A is a separable extension of k iff A is a separable algebra and iff T is nondegenerate (the bilinear form $A\times A\rightarrow k$ is defined by $(x, y)\longmapsto T(xy)$ as usual). But in case where an algebra A is not necessarily a field, the situation is more complicated.

Before stating Higman's result, let us fix the algebra A with several assumptions. Throughout this paper unless otherwise specified let k be a field with arbitrary characteristic and A a finite dimensional Frobenius k-algebra with nondegenerate associative bilinear from $\phi: A\times A\rightarrow k$ with fixed dual bases $\{a_i\}$, $\{b_i\}$ (i.e. $\phi(a_i, b_j)=\delta_{ij}$) $(i, j=1, \cdots, n)$.

Now Higman's result is as follows.

THEOREM 1 (Higman). *The following are equivalent.*

(i) *A is a separable k-algebra.*

(ii) *There exists a nondegenerate associative bilinear from $\phi: A\times A\rightarrow k$ with dual bases $\{a_i'\}$, $\{b_i'\}$ such that $\Sigma b_i' x a_i'=1$ for some $x\in A$.*

It should be noted that under the same situation of the theorem $\Sigma b_i' x a_i'$ is always in the center of A for any $x\in A$ but $\Sigma a_i' x b_i'$ is not (we shall give a simple example later). It seems natural to ask that what $\Sigma a_i' x b_i'=1$ does imply. To this question we obtained the following result.

THEOREM 2. *Let $\{a_i\}$, $\{b_i\}$ be dual bases with respect to ϕ. Then the following are equivalent.*

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