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ON THE MULTIPLICATIVE PARTITION FUNCTION

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1. Introduction.

Let n be a positive integer. A *multiplicative partition* of the number n is a representation of n as the product of any number of integers that are greater than 1. Thus

$$24=2\cdot 12=3\cdot 8=4\cdot 6=2\cdot 2\cdot 6=2\cdot 3\cdot 4=2\cdot 2\cdot 2\cdot 3$$

has 7 multiplicative partitions (cf. the table annexed at the end of this paper). Let us denote the number of mutiplicative partitions of n by $X(n)$, namely

$$X(n)=\sum_{n=2^{l_2}3^{l_3}4^{l_4}\ldots,\ l_2,\, l_3,\, l_4,\ldots\geqq 0} 1 \qquad (n>1)\,;$$

$X(1)$ is defined to be 1. This arithmetical function, we call it the multiplication partition function, was introduced by MacMahon [6] who noted that the function $X(n)$ has a generating function

$$G(s)\overset{\text{def}}{=}\sum_{n=1}^{\infty}X(n)n^{-s}=\prod_{m=2}^{\infty}(1-m^{-s})^{-1}\,, \qquad \mathrm{Re}\, s>1\,. \tag{1}$$

Making use of this relation, Oppenheim [7], [8] found an asymptotic formula

$$\sum_{n\leqq x}X(n)=\frac{x\, e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}\left\{1+\sum_{k=1}^{N-1}\frac{\varepsilon_k}{(\log x)^{k/2}}+O_N\left(\frac{1}{(\log x)^{N/2}}\right)\right\},$$

where the ε_k are certain constants, for each N and all large x. He also obtained a better approximation

$$\sum_{n\leqq x}X(n)=x\sum_{k=0}^{\infty}d_k\frac{I_{k+1}(2\sqrt{\log x})}{\sqrt{\log x}^{\,k+1}}+O\left(x\frac{e^{\sqrt{\log x}}}{(\log x)^{3/8}}\right) \tag{2}$$

to the sum $\sum_{n\leqq x}X(n)$, where the $I_k(x)$ are modified Bessel functions, and the numbers d_k are the coefficients in the Taylor expansion

$$\frac{G(s)}{s}e^{-1/(s-1)}=\sum_{k=0}^{\infty}d_k(s-1)^k, \qquad |s-1|<\frac{1}{2}\,. \tag{3}$$

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