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# ON $M$-RECURSIVELY SATURATED MODELS OF ARITHMETIC

By

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**Introducaion.**

In [6], C. Smorynski investigated the properties of models of arithmetic using the notions of recursive saturation and short recursive saturation. In this paper, we shall generalize these notions and obtain new isomorphism criteria (Theorems A and B) and embeddability criteria (Theorems D and E) for countable models of arithmetic.

Throughout, $\mathcal{PA}$ denotes Peano arithmetic with the induction schema for all formulas in some finite language $L\supseteqq\{0, ', +, \cdot\}$. $\Delta_0$ denotes the set of all quantifier bounded formulas in $L$. Let $M$ and $N$ be countable models of $\mathcal{PA}$ with $M\subseteqq N$. We say $N$ is *$M$-recursively saturated* (*$M^s$-recursively saturated*) if $N$ realizes every (short) type $\tau$ which is $\Delta_1$ on $\mathbf{HF}_M$, where $\tau$ may contain countably many parameters from $M$. It can be easily shown that $M$-recursive saturation ($M^s$-recursive saturation) corresponds with (short) recursive saturation, if $M=\langle\omega; 0, ', +, \cdot\rangle$. For $A\subseteqq|N|$, $Df(N, A)$ denotes the set of all elements in $N$ which are definable in $N$ using parameters from $A$. We put:

$Th_M(N)=\{\phi(c_{a_1}, \cdots, c_{a_n}): a_1, \cdots, a_n\in|M| \text{ and } N\models\phi(c_{a_1}, \cdots, c_{a_n})\}$,

$Th_M^{\Delta_0}(N)=\{\phi(c_{a_1}, \cdots, c_{a_n}): \phi\in\Delta_0, a_1, \cdots, a_n\in|M| \text{ and } N\models\phi(c_{a_1}, \cdots, c_{a_n})\}$,

$SS_M^{\Delta_0}(N)=\{X\cap|M|$ : $X$ is a subset of $|N|$ which is definable in $N$ using a $\Delta_0$-formula with parameters from $|N|\}$.

Our main results of this paper are as follows:

THEOREM A. *Suppose that $N_1$ and $N_2$ are $M$-recursively saturated countable models of $\mathcal{PA}$ such that $Th_M(N_1)=Th_M(N_2)$ and $SS_M^{\Delta_0}(N_1)=SS_M^{\Delta_0}(N_2)$. Then there is an isomorphism $f$: $N_1\to N_2$ which is identical on $M$.*

THEOREM B. *Suppose that $N_1$ and $N_2$ are $M^s$-recursively saturated models of $\mathcal{PA}$ such that $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$ and $SS_M^{\Delta_0}(N_1)=SS_M^{\Delta_0}(N_2)$. Suppose that both $N_1$ and $N_2$ are cofinal extensions of $M$. Then there is an isomorphism $f$: $N_1\to N_2$*

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