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ON THE DIFFERENCE $f^3(x)-g^2(x)$

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In 1965 H. Davenport [2] proved that if $f(x)$, $g(x)$ are polynomials in x with arbitrary real or complex coefficients, then we have either

$$f^3(x)-g^2(x)=0 \qquad \text{identically,}$$

or

$$\deg(f^3(x)-g^2(x)) \geqq \frac{1}{2}\deg f(x)+1\,. \tag{1}$$

It is known that for some pairs of polynomials $f(x)$, $g(x)$ the equality holds in (1). Clearly, we have for such pairs of polynomials

$$\deg f(x)=2k\,, \qquad \deg g(x)=3k$$

with some integral $k\geqq 1$. Indeed, if

$$f(x)=x^2+2\,, \qquad g(x)=x^3+3x\,,$$

then

$$f^3(x)-g^2(x)=3x^2+8\,,$$

and if

$$f(x)=x^4+2x\,, \qquad g(x)=x^6+3x^3+\frac{3}{2}\,,$$

then

$$f^3(x)-g^2(x)=-x^3-\frac{9}{4}\,.$$

Some other examples of pairs of polynomials $f(x)$, $g(x)$ of higher degrees satisfying the condition

$$\deg(f^3(x)-g^2(x))=\frac{1}{2}\deg f(x)+1 \tag{2}$$

are given by B. J. Birch, S. Chowla, Marshall Hall, Jr. and A. Schinzel [1]. They have shown in fact that

$$f(x)=x^6+4x^4+10x^2+6\,,$$

$$g(x)=x^9+6x^7+21x^5+35x^3+\frac{63}{2}x\,,$$

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