TSUKUBA J. MATH.
Vol. 6 No. 2 (1982). 177–186

ON ARONSZAJN TREES WITH A NON-SOUSLIN BASE

By

Masazumi HANAZAWA

§1. Introduction.

A tree is a partially ordered set $(T, <_T)$ with the property that for every element $x \in T$, $\hat{x}=\{y \in T : y <_T x\}$ is well-ordered by $<_T$. The order type of $\hat{x}$ is then an ordinal, which is called the height of x, $ht(x)$. When a subset of a tree is totally ordered by $<_T$, it is called a chain. When a subset of a tree has no comparable elements, it is called an antichain. We deal with only ω_1-trees which have cardinality ω_1, whose α-th level $T_\alpha=\{x \in T : ht(x)=\alpha\}$ is countable for every countable ordinal α, and which have additionally certain minor properties. An ω_1-tree T is said to be non-Souslin if every uncountable subset of T contains an uncountable antichain. A non-Souslin tree has clearly no uncountable chain and nevertheless for every countable ordinal α, the α-th level T_α is non-empty. This notion was introduced in Baumgartner [1]. The first example of a non-Souslin tree is the special Aronszajn tree which was given by Aronszajn (see Kurepa [5]). A special Aronszajn tree is characterized by Q-embeddability that means the existence of an order preserving function $f : T \rightarrow Q$. An R-(a fortiori, Q-) embeddable tree is always non-Souslin. Other examples of non-Souslin trees are found in Baumgartner [1], Hanazawa [2], [3] and Shelah [6]. Except for only one, the properties characterizing them are given as modifications of R-embeddability. The exception is the one given in [3], which has a non-Souslin base of cardinality ω_1. A non-Souslin base is a family F of uncountable antichains satisfying that whenever S is an uncountable subset of the tree T, there is an element A of F such that for every $x \in A$, there is $y \in S$ satisfying $x \leqq_T y$. Notice that a non-Souslin tree has always a non-Souslin base of cardinality 2^{ω_1}. We call a tree with a non-Souslin base of cardinality less than 2^{ω_1} an NSB-tree.

In this paper we discuss about NSB-trees, mainly to show that the property NSB is independent of R-embeddability. We first observe (in theorem 1) that under the standard set theory ZFC alone, even the existence of NSB-trees can

Received February 3, 1982.
This research was partially supported by Grant-in Aid for Scientific Research (No. 56340009), Ministry of Education.