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ON OBSERVABLE AND STRONGLY OBSERVABLE HOPF IDEALS

By

Hiroshi SHIGANO

Let $\mathfrak{G}=\mathfrak{Sp}_k A$ be an affine group scheme over a ground field k, $\mathfrak{H}=\mathfrak{Sp}_k A/I$ a closed subgroup scheme and $\bar{k}$ denote an algebraic closure of k. When $\mathfrak{G}$ is algebraic and $\mathfrak{G}(k)$ is dense in $\mathfrak{G}(\bar{k})$, Bialynichi-Birula, Hochschild and Mostow defined in their paper [1] that $\mathfrak{H}(k)$ is observable in $\mathfrak{G}(k)$ if every finite dimensional rational $\mathfrak{H}(k)$-module can be embedded in a rational $\mathfrak{G}(k)$-module as a $\mathfrak{H}(k)$-module, and showed that one of its necessary and sufficient conditions is $\mathfrak{G}(\bar{k})/\mathfrak{H}(\bar{k})$ being a quasi-affine variety. When $\mathfrak{G}$ is algebraic and reduced, and $k=\bar{k}$, Cline, Parshall and Scott defined in their paper [2] that $\mathfrak{H}(k)$ is strongly observable in $\mathfrak{G}(k)$ if every rational $\mathfrak{H}(k)$-module N can be embedded in a rational $\mathfrak{G}(k)$-module M as a $\mathfrak{H}(k)$-module such that $N^{\mathfrak{H}(k)}=M^{\mathfrak{G}(k)}$, and showed that one of its necessary and sufficient conditions is $\mathfrak{G}(k)/\mathfrak{H}(k)$ being an affine variety. Since these concepts are the representation-theoretic ones, we can extend them to a Hopf ideal of an arbitrary (not necessarily commutative) Hopf algebra. In this paper, we characterize a strongly observable Hopf ideal of an arbitrary Hopf algebra; a Hopf ideal I of a Hopf algebra A is strongly observable if and only if A is an injective A/I-comodule. This result and the one of M. Takeuchi [10] give the following equivalent characterizations of a strongly observable k-subgroup $\mathfrak{H}$ of an affine k-group $\mathfrak{G}$; (1) $\mathcal{O}(\mathfrak{G})$ is an injective k-$\mathfrak{H}$-module, (2) $\mathfrak{H}$ is exact in $\mathfrak{G}$ and (3) $\widetilde{\widetilde{\mathfrak{G}/\mathfrak{H}}}$ is affine. This is a generalization of a main theorem in [2]. In case of observable Hopf ideals, we do not have such a general characterization except the case when Hopf algebras are commutative. So we can extend some results in [1] to affine k-groups. One of them is that $\widetilde{\widetilde{\mathfrak{G}/\mathfrak{H}}}$ is quasi-affine if and only if I is observable in A and $R=A\square_{A/I}k$ contains a simple left coideal M such that $M\subset\sqrt{\mathfrak{A}}$ for any left coideal-ideal $\mathfrak{A}$ in R, where $\square$ denotes the cotensor product (see the first section).

Section 1 contains some preliminary results on cotensor products of comodules and injective comodules. It also contains one main result; for a Hopf ideal I of an arbitrary Hopf algebra A, A is an injective A/I-comodule if and only if A is an injective cogenerator for the category of A/I-comodules (1.6). Section 2 gives a characterization of a strongly observable Hopf ideal. Section 3 gives some results

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