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# ASYMPTOTIC RISK COMPARISON OF IMPROVED ESTIMATORS FOR NORMAL COVARIANCE MATRIX

By

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Asymptotic risks of the empirical Bayes estimators $\hat{\Sigma}_H$ by Haff [5] for a covariance matrix $\Sigma$ in a $p$-dimensional normal distribution are computed and compared with that of James and Stein's minimax estimators $\hat{\Sigma}_{JS}$. For $p\geqq 6$, it is shown that $\hat{\Sigma}_{JS}$ are always better than $\hat{\Sigma}_H$ asymptotically, though the leading terms are the same. New estimators which dominate $\hat{\Sigma}_{JS}$ for some $\Sigma$ in any $p$ asymptotically are proposed. Some numerical comparisons are given. Exact risks for ordinary estimators $\hat{\Sigma}_O$ and minimax estimators $\hat{\Sigma}_{JS}$ are also computed and compared with asymptotic ones for which the approximations are shown to be excellent.

## 1. Introduction

Let $S$ have a Wishart distribution with unknown scale matrix $\Sigma$ and $n$ degrees of freedom, for which we shall write $S: W_p(n, \Sigma)$ and assume $n>p+1$. Let $\hat{\Sigma}$ be an estimator of $\Sigma$. The loss function is taken to be

$$L_1(\hat{\Sigma}, \Sigma)=\mathrm{tr}\,\hat{\Sigma}\Sigma^{-1}-\log|\hat{\Sigma}\Sigma^{-1}|-p \tag{1.1}$$

or

$$L_2(\hat{\Sigma}, \Sigma)=\frac{1}{2}\mathrm{tr}(\hat{\Sigma}\Sigma^{-1}-I)^2. \tag{1.2}$$

The $L_1$ loss is equivalent to the likelihood ratio statistic for testing the hypothesis $\Sigma=\Sigma_0$ against all alternatives. The $L_2$ loss can also be used as a test statistic for the same problem as in Nagao [10]. The factor 1/2 in the $L_2$ loss is not essential. However we wish to retain it, since $L_1$ loss tends to $\mathrm{tr}(\hat{\Sigma}\Sigma^{-1}-I)^2/2$, when $\hat{\Sigma}$ is close to $\Sigma$. The risk function is given by $R_i(\hat{\Sigma}, \Sigma)=E[L_i(\hat{\Sigma}, \Sigma)]$ for $i=1$ or 2. Haff [5] proved that among the scalar multiples of $S$, the best estimator under $L_1$ is $\hat{\Sigma}_O^{(1)}=S/n$ and that under $L_2$ it is given by $\hat{\Sigma}_O^{(2)}=S/(n+p+1)$, which we call ordinary estimators. Then he considered the posterior mean of $\Sigma$ for a prior distribution $W_p[n', (\gamma C)^{-1}]$ for $\Sigma^{-1}$ with unknown scalar $\gamma>0$ and known p. d. matrix

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