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A DIFFERENTIAL GEOMETRIC CHARACTERIZATION OF HOMOGENEOUS SELF-DUAL CONES

(Dedicated to Professor K. Murata on his sixtieth birthday)

By

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In this note we give a differential geometric characterization of self-dual cones among affine homogeneous convex domains not containing any full straight line. Let Ω be an affine homogeneous convex domain in an n-dimensional real vector space V^n. Then Ω admits an invariant volume element

(1) $$v=\phi dx^1\wedge\cdots\wedge dx^n$$

and the canonical bilinear form defined by

(2) $$g=\sum_{i,j}\frac{\partial^2\log\phi}{\partial x^i\partial x^j}dx^i dx^j$$

is positive definite and so gives an invariant Riemannian metric on Ω, where $\{x^1,\cdots,x^n\}$ is an affine coordinate system on V^n [5]. In an affine coordinate system $\{x^1,\cdots,x^n\}$ the components of the Riemannian connection Γ and the Riemannian curvature tensor R for g are expressed as follows

(3) $$\Gamma^i{}_{jk}=\frac{1}{2}\sum_p g^{ip}\frac{\partial^3\log\phi}{\partial x^j\partial x^k\partial x^p},$$

(4) $$R^i{}_{jkl}=\sum_p(\Gamma^i{}_{pk}\Gamma^p{}_{jl}-\Gamma^i{}_{pl}\Gamma^p{}_{jk}),$$

where $g_{ij}=\dfrac{\partial^2\log\phi}{\partial x^i\partial x^j}$ and $\sum_p g^{ip}g_{pj}=\delta^i{}_j$ (Kronecker's delta). Since $\dfrac{1}{2}\sum_p g^{ip}\dfrac{\partial^3\log\phi}{\partial x^j\partial x^k\partial x^p}$ defines a tensor field on Ω, we denote this tensor field by the same letter Γ.

An open convex set Ω in V^n is called a cone with vertex o if $o+\lambda(x-o)\in\Omega$ for all $x\in\Omega$ and $\lambda>0$. An open convex cone Ω with vertex o is said to be a self-dual cone if V^n admits an inner product $\langle\ ,\ \rangle$ such that

(i) $\langle x-o, y-o\rangle>0$ for all $x, y\in\Omega$;

(ii) if $x\in V^n$ is a vector such that $\langle x-o, y-o\rangle\geqq 0$ for all $y\in\bar{\Omega}$ then $x\in\bar{\Omega}$, where $\bar{\Omega}$ is the closure of Ω in V^n.

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