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ON TRIVIAL EXTENSIONS WHICH ARE QUASI-FROBENIUS ONES

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Recently Y. Kitamura has characterized a trivial extension which is a Frobenius extension in [2]. In this paper we characterize a trivial extension which is a quasi-Frobenius extension.

Let R be a ring with an identity and M an (R, R)-bimodule. The trivial extension $S=(R, M)$ of R by M is the direct sum of additive groups R and M with the multiplication $(r_1, m_1)(r_2, m_2)=(r_1r_2, r_1m_2+m_1r_2)$ for $(r_i, m_i)\in S$. S is a ring containing R with the identification $r\rightarrow(r, 0)$ for $r\in R$. Let $*S$ be the dual space of S as a left R-module. Then $*S$ is isomorphic to the direct sum of R and $*M=\mathrm{Hom}\,({}_RM, {}_RR)$: $*S=[R, *M]$. The action of an element $[a, h]\in *S$ on S is given by $[a, h]((r, m))=ra+h(m)$ for $(r, m)\in S$. $*S$ has the structure of an (S, R)-bimodule. This is given by $(r, m)[a, h]=[ra+h(m), rh]$ and $[a, h]r=[ar, hr]$ for $(r, m)\in S, [a, h]\in *S$ and $r\in R$.

Following to [3] a ring extension S over R is called a left quasi-Frobenius extension when S is left R-finitely generated projective and a direct summand of a finite direct sum of $*S$ as an (S, R)-bimodule.

Let S be the trivial extension of R by M, and assume that S is a left quasi-Frobenius extension of R. Then there exist (S, R)-homomorphisms $\Phi: S\rightarrow *S\oplus\cdots\oplus *S$ and $\Psi: *S\oplus\cdots\oplus *S\rightarrow S$ such that $\Psi\circ\Phi=1_S$. Let $\Phi((1, 0))=([a_1, h_1], \cdots, [a_n, h_n])$. Then it is easily seen that h_i is contained in $\mathrm{Hom}\,({}_RM_R, {}_RR_R)$ for all i. Next, we consider homomorphisms from $*S$ to S. Since S is left R-finitely generated projective, we have following isomorphisms

$$\begin{aligned}\mathrm{Hom}\,({}_S*S_R, {}_SS_R)&=\mathrm{Hom}\,({}_S\mathrm{Hom}\,({}_RS, {}_RR)_R, {}_SS_R)\\ &\cong\{\mathrm{Hom}\,(R_R, S_R)\otimes_RS\}^S\cong\{S\otimes_RS\}^S\end{aligned}$$

where $\{S\otimes_RS\}^S$ means the set of elements in $S\otimes_RS$ commuting to the elements of S. Explicitely, the correspondence is given by $\sum(s_1\otimes s_2)(f)=\sum s_1f(s_2)$ for $\sum s_1\otimes s_2\in\{S\otimes_RS\}^S$ and $f\in *S$. Let Ψ_i be the restriction of Ψ to i-th component of $*S\oplus\cdots\oplus *S$ and $\Sigma_j(b_{ij}, m_{ij})\otimes(c_{ij}, n_{ij})$ the corresponding element in $\{S\otimes_RS\}^S$. Then, for $[a, h]\in *S$, we have

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