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A NOTE ON SYMMETRY OF PERPENDICULARITY IN A G-SPACE WITH NONPOSITIVE CURVATURE

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1. Introduction.

Let $\mathfrak{R}$ be a G-space. We denote by $m(a, b)$, for $a, b\in\mathfrak{R}$, a midpoint so that $am(a, b)=m(a, b)b=ab/2$. The G-space $\mathfrak{R}$ has "*nonpositive curvature*" if every point p has a neighborhood $S(p, \gamma_{p})$, where $0<\gamma_{p}<\rho_{1}(p)$ (see [3] for definition of $\rho_{1}(p)$), such that for any three points a, b, c in $S(p, \gamma_{p})$ the relation $2m(a, b)m(a, c)\leq bc$ holds, and R has "*negative curvature*" if $2m(a, b)m(a, c)<bc$ when a, b, c are not on one segment. Because a G-space $\mathfrak{R}$ with nonpositive curvature is finite-dimensional according to V. N. Berestovskii [1], and hence $\mathfrak{R}$ has "*domain invariance*" (see [4] p. 16), the universal covering space $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ is straight by Busemann [3] p. 254. Moreover the spheres in $\overline{\mathfrak{R}}$ are convex. The straight line L in a G-space is called a "*perpendicular*" to the set M at f, if $f\in L\cap M$ and every point of L has f as a *foot* on M, i.e., $qf=qM$ for any $q\in L$. We say that perpendicularity between lines is *symmetric* if the following holds: if a straight line L is a perpendicular to a straight line G, then G is a perpendicular to L. We say that a set M of a G-space is *totally convex* if $p, q\in M$ implies that all geodesic curves from p to q are contained in M. If a closed set M of a G-space $\mathfrak{R}$ in which the spheres are convex is totally convex, then for each $p\in\mathfrak{R}$ there is a unique point $q\in M$ such that $pq=pM$. If the spheres of a straight G-space are convex, we denote by W_{p} the point set carring straight lines through $p\in K(q, \sigma):=\{r\, ;qr=\sigma\}$ but not through any point $p'\in S(q, \sigma)=\{r\, ;qr<\sigma\}$, which are called the supporting lines of $K(q, \sigma)$ at p. $K(q, \sigma)$ is "*differentiable at $p\in K(q, \sigma)$*" if no proper subset of the W_{p} decomposes the space.

In the present paper we prove

THEOREM 1. *Let $\mathfrak{R}$ be a G-space of nonpositive curvature. If the spheres in the universal covering space $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ are differentiable and if perpendicularity between lines is symmetric in $\overline{\mathfrak{R}}$, then for every closed totally convex set M in $\mathfrak{R}$ the map $\rho:\mathfrak{R}\rightarrow M$ defined by sending $p\in\mathfrak{R}$ to the foot of p on M is distance non-increasing. Further, if $pq=\rho(p)\rho(q)\neq 0$, then $S:=\bigcup_{0\leq t\leq 1}T(p_{t}, q_{t})$ is isometric to a trapezoid in a Minkowski plane, where $T(p_{t}, q_{t})$ is the point set carring the segment from $p_{t}\in T(p, \rho(p))$ to $q_{t}\in T(q, \rho(q))$ with $pp_{t}:p_{t}\rho(p)=qq_{t}:q_{t}\rho(q)=t:(1-t)$.*

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