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ZETA FUNCTIONS OF INTEGRAL GROUP RINGS OF METACYCLIC GROUPS

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Recently, Solomon has introduced a zeta function which counts sublattices of a given lattice over an order ([5]). Let us recall the definition of this zeta function. Let Σ be a (finite dimensional) semisimple algebra over the rational field $\boldsymbol{Q}$ or over the p-adic field $\boldsymbol{Q}_p$, and let Λ be an order in Σ. Λ is a $\boldsymbol{Z}$-order when Σ is a $\boldsymbol{Q}$-algebra, while Λ is a $\boldsymbol{Z}_p$-order when Σ is a $\boldsymbol{Q}_p$-algebra, where $\boldsymbol{Z}_p$ is the ring of p-adic integers. Throughout this paper, p stands for a rational prime and the subscript p indicates the p-adic completion.

Let V be a finitely generated left Σ-module, and let L be a full Λ-lattice in V. Solomon's zeta function is defined as

$$\zeta_\Lambda(L; s)=\sum_N (L:N)^{-s},$$

where the sum $\sum_N$ extends over all full Λ-sublattices N in L, $(L:N)$ denotes the index of N in L and s is a complex variable. We shall omit the subscript Λ and write $\zeta(L; s)$, unless there is danger of confusion. When Σ is a field K and L is the ring of integers in K, $\zeta_K(L; s)$ is the classical Dedekind zeta function, and we shall denote this by $\zeta_L(s)$.

We denote by C_n the cyclic group of order n. The explicite form of $\zeta(\boldsymbol{Z}G; s)$ has been given for each of the cases $G=C_p$ and C_{p^2} ([4], [5]).

Let q be a prime and let n be a square-free integer coprime to q. Let $C_n\cdot C_q$ be the semidirect product of C_n by C_q in which C_q acts faithfully on the subgroup C_p of C_n for every $p|n$. The aim of this paper is to give an explicit form of $\zeta(\boldsymbol{Z}(C_n\cdot C_q); s)$. We shall use the method introduced in [1].

§1. Let Λ be a $\boldsymbol{Z}$-order in a semisimple $\boldsymbol{Q}$-algebra Σ, and let $\mathfrak{M}$ be a maximal $\boldsymbol{Z}$-order containing Λ. Denote by S the set of primes p for which $\Lambda_p\neq\mathfrak{M}_p$. Since the zeta function satisfies the Euler product identity ([5]), we have

$$\zeta_\Lambda(\Lambda; s)=\zeta_{\mathfrak{M}}(\mathfrak{M}; s)\times\prod_{p\in S}\frac{\zeta_{\Lambda_p}(\Lambda_p; s)}{\zeta_{\mathfrak{M}_p}(\mathfrak{M}_p; s)}. \tag{1.1}$$

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