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# NON-STANDARD REAL NUMBER SYSTEMS WITH REGULAR GAPS

By

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The purpose of this paper is to show that if an enlargement $^*\boldsymbol{M}$ of the universe $\boldsymbol{M}$ is saturated, then the non-standard real number system $^*\boldsymbol{R}$ has a regular gap and the uniform space $(^*\boldsymbol{R}, E[L(1)])$ is not complete.

Our notions and terminologies follow the usual use in the model theory. Let $\boldsymbol{G}=\langle G, +, <\rangle$ be a first order structure which satisfies

(a) the axioms of ordered abelian groups,

(b) the axioms of dense linear order.

(i. e. $\langle G, +, <\rangle$ is an ordered abelian group and $\langle G, <\rangle$ is a densely ordered set.) A Dedekind cut $(X, Y)$ in $\boldsymbol{G}$ is said to be a *gap* if $sup(X)$ $(inf(Y))$ does not exist. A gap $(X, Y)$ is said to be *regular* if, for all $e$ in $G_+$ $(=\{g\in G\,;\, g>0\})$, $X+e\neq X$.

THEOREM. *Suppose that* $\boldsymbol{G}$ *is saturated. Then,* $\boldsymbol{G}$ *has a regular gap. Moreover,* $\boldsymbol{G}$ *has* $2^{\kappa}$*-th regular gaps, where* $\kappa$ *is the cardinality of* $G$.

PROOF. Since $\boldsymbol{G}$ is saturated, the coinitiality of $G_+$ is $\kappa$. Let $\langle g_\alpha \mid \alpha<\kappa\rangle$ be an enumeration of $G$ and let $\langle e_\alpha \mid \alpha<\kappa\rangle$ be a strictly decreasing coinitial sequence in $G_+$. By the induction on $\alpha<\kappa$, we shall define a set $\{I(x_u, y_u)\,;\, u\in {}^{\alpha}2\}$ of open intervals in $\boldsymbol{G}$ such that

(1) $I(x_u, y_u)\neq\varnothing$ for all $u$ in ${}^{\alpha}2$,

(2) $y_u-x_u<e_\alpha$ for all $u$ in ${}^{\alpha}2$,

(3) $g_\alpha\notin I(x_u, y_u)$ for all $u$ in ${}^{\alpha}2$,

(4) $I(x_u, y_u)\cap I(x_v, y_v)=\varnothing$ for all distinct elements $u$, $v$ in ${}^{\alpha}2$,

(5) for all $\beta<\alpha$, for all $v\in{}^{\beta}2$ and for all $u\in{}^{\alpha}2$, if $v\subset u$, then $I(x_v, y_v)\supset I(x_u, y_u)$.

The construction is as follows:

(Case 1) $\alpha=0$.

This case is obvious.

(Case 2) $\alpha=\beta+1$ for some $\beta$.

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