ON THE LENGTH OF PROOFS IN FORMAL SYSTEMS

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§ 0. Introduction.

This paper is concerned with an aspect of lengths of proofs in formal systems. For a first order system $T$, by $\frac{k}{T} A$, we will mean that $A$ is provable in $T$ with at most $k$ applications of rules of inference. Let $PA^*$ be a system for Peano arithmetic with only one function symbol $S$ for successor and two predicate symbols which represent addition and multiplication respectively.

In [2] R. Parikh proved:

1. For any given formula $A$ and natural number $k$, it is decidable whether $\frac{k}{PA^*} A$ holds or not.

2. $\frac{\forall x A(x)}{PA^*}$ iff there is a $k$ such that $(\forall n) \frac{k}{PA^*} A(\overline{n})$.

In this paper we shall prove an analogue of (2) for systems which have a finite number of function symbols and a finite number of axiom schemata, and are complete with respect to formulas in Presburger arithmetic i.e. formulas which have only $S$, $+$, $=$ other than logical symbols.

Let $T$ be any one of such systems. By $T_k$ we mean the subsystem of $T$ which has only axioms containing at most $k$ occurrences of bound variables and critical explicit terms (these will be defined in §1). Now our claim is:

$\frac{\forall x A(x)}{T}$ iff there is a $k$ such that $(\forall n) \frac{k}{T_k} A(\overline{n})$.

This implies Parikh's result (2), for it is easy to see that $(\forall n) \frac{k}{PA^*} A(\overline{n})$ iff there exists $r$ such that $(\forall n) \frac{k}{PA_r^*} A(\overline{n})$.

T. Yukami has proved an analogous result as (2) for a system of natural numbers with two function symbols for successor and addition, with one predicate symbol which represents multiplication.

This system has as its axioms not only usual ones, but also all valid equations $t=u$. Since his system does not fall under a system with finitely many axiom schemata, we cannot treat his system by the method in this paper.

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