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ON THE DEFINITION OF THE WAVE FRONT SET OF A DISTRIBUTION

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§ 1. Introduction.

Gabor [2] has indicated an invariant definition of the wave front set of a distribution. His definition in a slightly modified form is as follows. Let X be a manifold, $(x_0, \xi_0)\in T^*X\backslash 0$ and $u\in\mathscr{D}'(X)$. Then we say that (x_0, ξ_0) is in the complement of the wave front set $WF(u)$ of u, if and only if there exists a conic open neighborhood $\Gamma\subset T^*X\backslash 0$ of (x_0, ξ_0) such that for every compact set F of real-valued functions $f\in C^\infty(X)$ and every $g\in C_0^\infty(X)$ with $(x, df(x))\in\Gamma$ for $x\in\mathrm{supp}\, g$, we have for every integer $k\geq 0$, when $\tau\rightarrow\infty$,

$$\langle e^{-i\tau f}g, u\rangle=O(\tau^{-k})$$

uniformly in $f\in F$.

In terms of local coordinates an equivalent definition of $WF(u)$ has been given using the Fourier transform [1], [3]. Let X be identified with an open subset of R^n and T^*X with $X\times R^n$. Then $(x_0, \xi_0)\notin WF(u)$ if and only if there exists a function $\chi\in C_0^\infty(X)$ with $\chi(x_0)\neq 0$ and a conic neighborhood $\Xi\subset R^n\backslash 0$ of ξ_0 such that for every integer $k\geq 0$ we have

$$\langle e^{-i\langle x,\xi\rangle}\chi, u\rangle=O(|\xi|^{-k}),$$

when $|\xi|\rightarrow\infty$ in Ξ.

In the first definition we test the distribution u by oscillatory test functions $e^{-i\tau f}g$ with arbitrary phase functions $f\in F$, while in the second we need only test u by a single oscillating function $e^{-i\langle x,\xi\rangle}\chi$ with the linear phase $\langle x, \xi\rangle$ depending on a parameter ξ. This suggests that testing the distribution u by a single oscillatory test function of the form $e^{-i\tau\psi(x,\sigma)}\chi(x)$ containing a parameter σ will suffice under a suitable condition on the dependence of the phase function $\psi(x, \sigma)$ on the parameter σ. The purpose of this paper is to give a sufficient condition. The phase function $\psi(x, \sigma)$ is allowed to be nonlinear and this will probably be useful in the calculus of wave front sets. It is possible to prove the result in this paper using the theory of Fourier integral operators. However we give here an elementary

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