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A NOTE ON PROVABLE WELL-ORDERINGS IN FIRST ORDER SYSTEMS WITH INFINITARY INFERENCE RULES

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In [4], Shirai refined the well-known result on the provable well-orderings of the pure number theory in Gentzen [2] as follows:

Transfinite induction up to α is provable by only the induction rules (VJ-inferences) with induction formulae which have at most ρ quantifiers if and only if α represents an ordinal which is smaller than ω_{ρ}, where $\omega_0=\omega, \omega_{n+1}=\omega^{(\omega_n)}$.

In this paper we refine the result of Schütte [3] corresponding to the result of unprovability in Gentzen [2] according to the spirit of Shirai [4] as follows:

If P is an infinitary proof of the transfinite induction up to α, i.e., $\forall x(\forall y\prec xXy\supset Xx)\longrightarrow X\alpha$, and the ordinal number of P is smaller than $\omega\cdot\beta$, then α represents an ordinal which is smaller than $B(\beta, n)$, where n is the maximum length of the sequences of mutually regulating occurences of quantifiers in the cut formulae in P and B is the function defined at the beginning of § 2 below.

In § 1 we introduce our syntax and prove the reduction theorem which is a refined version of the Reduktionssatz of Schütte [3]. In § 2 we prove the upper bound theorem which is the main result of the present note by the reduction theorem and two lemmata on the structure of the derivations of transfinite induction. As a corollary of the upper bound theorem we prove a part of the Shirai's result, i.e., if the transfinite induction up to α is provable by only the inductions whose induction formulae have at most ρ quantifiers and $\rho\geq 1$, then α is smaller than ω_{ρ}.

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§ 1. Syntax and the Reduction Theorem

We use only two logical symbols $\forall, \supset$. Let $\mathcal{L}$ be a first order language which has a binary predicate constant $\prec$ at least. Let $\mathfrak{A}$ be a structure of $\mathcal{L}$ in which $\prec$ is a well-ordering of $|\mathfrak{A}|$. We introduce a new constant c_a for each element $a\in|\mathfrak{A}|$. We understand that if $a\neq b$, then c_a and c_b are different symbols. We de-

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