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TIME DECAY ESTIMATES OF SOLUTIONS TO THE MIXED PROBLEM FOR HEAT EQUATIONS IN A HALF SPACE

By

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1 Introduction

The Cauchy problem for the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & t > 0,\ x \in \mathbf{R}^n, \\ u|_{t=0} = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

has a solution

$$u(t,x) = \frac{1}{\sqrt{4\pi t}^n} \int_{\mathbf{R}^n} e^{-|x-y|^2/4t} u_0(y)\, dy \tag{1.2}$$

which has the following three estimates for $t > 0$

$$\|u(t)\|_{L^\infty} \le \frac{c_n}{t^{n/2}} \|u_0\|_{L^1}, \tag{1.3}$$

$$\|u(t)\|_{L^p} \le c_{n,p} \|u_0\|_{L^p}, \quad 1 \le p \le \infty, \tag{1.4}$$

and

$$\|u(t)\|_{L^p} \le \frac{c_{n,p,q}}{t^{(n/2)(q^{-1}-p^{-1})}} \|u_0\|_{L^q}, \quad 1 \le q < p \le \infty, \tag{1.5}$$

where

$$\|u\|_{L^p} = \left(\int_{\mathbf{R}^n} |u(x)|^p\, dx \right)^{1/p}.$$

(1.3) and (1.4) follow immediately from (1.2). We can derive (1.5) from (1.3) and (1.4) by use of interpolation (see Proposition 2.1 below).

Dirichlet and Neumann problem in a half space $\mathbf{R}^n_+ = \{x = (x', x_n), x' \in \mathbf{R}^{n-1}, x_n > 0\}$ has a solution respectively as

$$u_D(t,x) = \frac{1}{\sqrt{4\pi t}^n} \int_{\mathbf{R}^n_+} [e^{-(|x'-y'|^2-(x_n+y_n)^2)/4t} - e^{-(|x'-y'|^2-(x_n-y_n)^2)/4t}] u_0(y)\, dy \tag{1.6}$$

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