Homotopy classification theorem in algebraic geometry

Dedicated to Professor Yoshie Katurada on her sixtieth birthday

By Toshiyuki MAEBASHI

Introduction. Let X be *a finite CW complex*. We denote by $K(X)$ the Grothendieck group of the classes of complex vector bundles over X. We further write Z, B_U for the integers with the discrete topology, the classifying space of the infinite unitary group respectively. Then the K-theoretic version of the homotopy classification theorem is given by the statement of the existence of a natural bijection:

$$K(X) \cong [X, B_U \times Z]$$

where $[X, B_U \times Z]$ denotes the set of homotopy classes of maps of X into $B_U \times Z$.

The objective of this paper is to present an algebro-geometric analogue to the above-mentioned theorem. We consider *a non-singular reduced affine k-scheme for an algebraically closed field k*, instead of a finite CW complex. Let X be a k-scheme of this kind. We write $K(X)$ for the Grothendieck group of the classes of coherent O_X-Modules. Let $G_{n,n}$ be the Grassmannian k-scheme of n-planes in affine $2n$-space A_k^{2n} where n ranges over the positive integers. Then there are natural closed immersions: $G_{n,n} \longrightarrow G_{l,l}$ for $l > n$. We denote by B_k the direct limit of $G_{n,n}$ in the category of geometrical k-spaces. Consider morphisms $f, g: X \longrightarrow B_k \times Z$. We define $f \sim g$ if and only if f is connected with g by a finite chain of rational homotopies. A class by the equivalence relation $\sim$ will be called *a rational homotopy class*. We write $[X, B_k \times Z]_{\mathrm{rat}}$ for the set of rational homotopy classes of k-morphisms: $X \longrightarrow B_k \times Z$. With these notations we have

Main Theorem. *There is a natural bijection*

$$K(X) \cong [X, B_k \times Z]_{\mathrm{rat}}\,.$$

Let X be an irreducible algebraic prescheme over an algebraically closed field k. Let Υ_n^m be the universal scheme vector bundle over $G_{n,m}$, i.e. the Grassmannian k-scheme of n-planes in affine $(m-n)$-space. We denote by p the natural projection: $\Upsilon_n^m \longrightarrow G_{n,m}$. We now state two theorems below which are used for the proof of the Main Theorem, because of their own interest.